Sommes télescopiques

Sommes doubles

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$$\sum_{k=0}^{n} \underbrace{(k+1)^2}_{a_{k+1}} - \underbrace{k^2}_{a_k} = (n+1)^2 - 0^2$$
$$= (n+1)^2$$

Principe $\displaystyle\sum_{k=0}^{n} a_{k+1} - a_k = \cancel{a_1} - a_0 + \cancel{a_2} - \cancel{a_1} + \cancel{a_3} - \cancel{a_2} + \dots$

$$+ \cancel{a_n} - \cancel{a_{n-1}}$$
$$+ a_{n+1} - \cancel{a_n}$$

$$= a_{n+1} - a_0$$

Preuve $\displaystyle\sum_{k=0}^{n} a_{k+1} - a_k = \sum_{k=0}^{n} a_{k+1} - \sum_{k=0}^{n} a_k$

$$= \sum_{k=1}^{n+1} a_k - \sum_{k=0}^{n} a_k$$

$$= \left(\cancel{\sum_{k=1}^{n} a_k}\right) + \boxed{a_{n+1} - \left[a_0 \right.} + \left(\cancel{\sum_{k=1}^{n} a_k}\right)\Bigg]$$

Example

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \,?$$

$n=1$ $\quad \frac{1}{2}$

$n=2$ $\quad \frac{1}{2}+\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$

$n=3$ $\quad \frac{2}{3}+\frac{1}{12}=\frac{9}{12}=\frac{3}{4}$

$n=4$ $\quad \frac{3}{4}+\frac{1}{20}=\frac{16}{20}=\frac{4}{5}$

$\frac{n}{n+1}$?

$$\frac{1}{k(k+1)} = \frac{a}{k} + \frac{b}{k+1} = \frac{a(k+1)+bk}{k(k+1)} = \frac{(a+b)k+a}{k(k+1)}$$

$$\begin{cases} a+b=0 \\ a=1 \end{cases} \Longleftrightarrow \begin{cases} a=1 \\ b=-1 \end{cases} \qquad \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right) = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}} + \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1} = \frac{n+1}{n+1} - \frac{1}{n+1} = \frac{n}{n+1}$$

simple : $\sum_{k=0}^{n} a_k$

Sommes doubles

$$\sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} a_{ij} = \sum_{i=1}^{n} \underbrace{\left(\sum_{j=1}^{m} a_{ij} \right)} = \sum_{j=1}^{m} \underbrace{\left(\sum_{i=1}^{n} a_{ij} \right)}$$

Visualisation

$$\begin{array}{ccccccc}
a_{11} & a_{12} & a_{13} & \cdots & a_{1m} & \longrightarrow & \sum_{j=1}^{m} a_{1j} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2m} & \longrightarrow & \sum_{j=1}^{m} a_{2j} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3m} & \longrightarrow & \sum_{j=1}^{m} a_{3j} \\
\vdots & \vdots & \vdots & \ddots & & & \vdots \\
a_{n1} & a_{n2} & a_{n3} & & a_{nm} & \longrightarrow & \sum_{j=1}^{m} a_{nj} \\
\downarrow & \downarrow & \downarrow & & \downarrow & & \\
\sum_{i=1}^{n} a_{i1} & \sum_{i=1}^{n} a_{i2} & \sum_{i=1}^{n} a_{i3} & & \sum_{i=1}^{n} a_{im} & &
\end{array}$$

Exemple

$$\sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} ij$$

$n = 2$ L

$m = 3$ C

	$j=1$	$j=2$	$j=3$
$i=1$	1	2	3
$i=2$	2	4	6

$\rightarrow 18$

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{m} ij \right) = \sum_{i=1}^{n} \left(i \sum_{j=1}^{m} j \right)$$

$$= \sum_{i=1}^{n} \left(i \times \frac{m(m+1)}{2} \right)$$

$$= \frac{m(m+1)}{2} \times \left(\sum_{i=1}^{n} i \right)$$

$$= \frac{m(m+1)}{2} \times \frac{n(n+1)}{2}$$

$n = 2$, $m = 3$

6×3

$= 18$

Cas particuliers

$$\sum_{1\le i,j\le n} a_{ij} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ij}\right)$$

Carré

$$\begin{matrix} a_{11} & a_{12} & — & a_{1n} \\ a_{21} & a_{22} & — & a_{2n} \\ | & | & \diagdown & | \\ a_{n1} & a_{n2} & — & a_{nn} \end{matrix}$$

$$\sum_{1\le i\le j\le n} a_{ij}$$

$$\begin{array}{c|cccc|l} & j=1 & j=2 & \cdots & j=n & \\ \hline i=1 & a_{11} & a_{12} & \cdots & a_{1n} & \leftarrow \sum_{j=1}^{n} a_{1j} \\ i=2 & \times & a_{22} & \cdots & a_{2n} & \leftarrow \sum_{j=2}^{n} a_{2j} \\ \vdots & \times & \times & \ddots & \vdots & \\ i=n & \times & \times & \times\ \cdots & a_{nn} & \\ \hline & \uparrow & \uparrow & & \uparrow & \\ & \sum_{i=1}^{1} a_{i1} & \sum_{i=1}^{2} a_{i2} & & \sum_{i=1}^{n} a_{in} & \end{array}$$

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{j} a_{ij}\right) = \sum_{i=1}^{n}\left(\sum_{j=i}^{n} a_{ij}\right)$$

À vous de jouer !

① Simplifier $\displaystyle\sum_{k=0}^{n} \frac{1}{(k+1)(k+3)}$

② Simplifier :

$$\sum_{1 \leq i \leq j \leq n} ij \qquad\qquad \sum_{1 \leq i, j \leq n} \min(i,j)$$