# Sommes télescopiques

# Sommes doubles (Exercices)

① Simplifier $\displaystyle\sum_{k=0}^{n} \frac{1}{(k+1)(k+3)}$

$$\frac{1}{(k+1)(k+3)} = \frac{a}{k+1} + \frac{b}{k+3} = \frac{a(k+3)+b(k+1)}{(k+1)(k+3)} = \frac{(a+b)k + (3a+b)}{(k+1)(k+3)}$$

$$\begin{cases} a+b=0 \\ 3a+b=1 \end{cases} \Longleftrightarrow \begin{cases} a+b=0 \\ 2a=1 \end{cases} \Longleftrightarrow \begin{cases} a=1/2 \\ b=-1/2 \end{cases} = \frac{1/2}{k+1} - \frac{1/2}{k+3}$$

$$\frac{1}{2}\sum_{k=0}^{n}\left(\frac{1}{k+1} - \frac{1}{k+3}\right) = \frac{1}{2}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \dots + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2} + \frac{1}{n+1} - \frac{1}{n+3}\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{n}\frac{1}{k+1} - \sum_{k=0}^{n}\frac{1}{k+3}\right] = \frac{1}{2}\left[\sum_{k=1}^{n+1}\frac{1}{k} - \sum_{k=3}^{n+3}\frac{1}{k}\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} + \cancel{\sum_{k=3}^{n+1}\frac{1}{k}} - \left(\cancel{\sum_{k=3}^{n+1}\frac{1}{k}} + \frac{1}{n+2} + \frac{1}{n+3}\right)\right] = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

② $\displaystyle\sum_{1 \le i \le j \le n} ij$

$n = 4$

| | $j=1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|
| $i=1$ | 1 | 2 | 3 | 4 |
| $2$ | × | 4 | 6 | 8 |
| $3$ | × | × | 9 | 12 |
| $4$ | × | × | × | 16 |

$$\sum_{i=1}^{j} ij$$

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{j} ij\right) = \sum_{j=1}^{n}\left(j\sum_{i=1}^{j} i\right)$$

$$= \sum_{j=1}^{n}\left(j\,\frac{j(j+1)}{2}\right)$$

$$= \frac{1}{2}\sum_{j=1}^{n}\left(j^3+j^2\right)$$

$$= \frac{1}{2}\sum_{j=1}^{n} j^3 + \frac{1}{2}\sum_{j=1}^{n} j^2$$

$$= \frac{n^2(n+1)^2}{8} + \frac{n(n+1)(2n+1)}{12}$$

$$\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{1 \le i,j \le n} \min(i,j)$$

| | $j=1$ | 2 | 3 | 4 | 5 | |
|---|---|---|---|---|---|---|
| $i=1$ | 1 | 1 | 1 | 1 | 1 | – 5 |
| 2 | 1 | 2 | 2 | 2 | 2 | – 9 |
| 3 | 1 | 2 | 3 | 3 | 3 | – 12 |
| $i=4$ → 4 | 1 | 2 | 3 | 4 | 4 | – 14 |
| 5 | 1 | 2 | 3 | 4 | 5 | – 15 |

$\sum = 55$

$$\sum_{i=1}^{n} \Bigg( \underbrace{\sum_{j=1}^{n} \min(i,j)}_{\displaystyle \sum_{j=1}^{i} j + \sum_{j=i+1}^{n} i} \Bigg)$$

$$\sum_{j=1}^{i} j + \sum_{j=i+1}^{n} i = \frac{i(i+1)}{2} + i(n-i) = -\frac{i^2}{2} + \left(n+\frac{1}{2}\right) i$$

$n-(i+1)+1 = (n-i)$ termes

$$-\frac{1}{2}\sum_{i=1}^{n} i^2 + \left(n+\frac{1}{2}\right)\sum_{i=1}^{n} i = -\frac{n(n+1)(2n+1)}{12} + \left(n+\frac{1}{2}\right)\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)(2n+1)}{6} = \sum_{k=1}^{n} k^2 \quad ??$$

$n=7$

$$\begin{array}{c|ccccccc}
 & j=1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
i=1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 1 & 2 & 3 & 3 & 3 & 3 & 3 \\
4 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\
5 & 1 & 2 & 3 & 4 & 5 & 5 & 5 \\
6 & 1 & 2 & 3 & 4 & 5 & 6 & 6 \\
7 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
 & \uparrow & \uparrow & \uparrow & & & & \\
 & 2(n-1)+1 & 2(n-2)+1 & 2(n-3)+1 & & & &
\end{array}$$

$\min(i,j)$

$$\sum_{k=1}^{n} k \times (2(n-k)+1) = \sum_{k=1}^{n} -2k^2 + (2n+1)k$$

$$= -\frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)(2n+1)}{2} = \frac{n(n+1)(2n+1)}{6}$$

# Exercices bonus

① Simplifier :

$$\sum_{k=1}^{n} \sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}$$

② Simplifier : $\displaystyle\sum_{1\leq i,j\leq n} \max(i,j)$

Solutions : martintroussard.fr