

Products (Exercises)

$$\begin{aligned}
 a) \quad \prod_{h=1}^n \frac{4^h}{h^2} &= \frac{\prod_{h=1}^n 4^h}{\prod_{h=1}^n h^2} = \frac{4^1 \times 4^2 \times 4^3 \times \dots}{1^2 \times 2^2 \times 3^2 \times \dots} = 4^{1+2+3+\dots} \\
 &= \frac{4^{\sum_{h=1}^n h}}{\left(\prod_{h=1}^n h\right)^2} \\
 &= \frac{4^{\frac{n(n+1)}{2}}}{(n!)^2}
 \end{aligned}$$

$$\begin{aligned}
b) \quad \prod_{k=1}^n (6k-3) &= \prod_{k=1}^n 3(2k-1) \\
&= 3^n \underbrace{\prod_{k=1}^n (2k-1)}_{= 1 \times 3 \times 5 \times \dots \times (2n-1)} \\
&= 3^n \frac{(2n-1)!}{\prod_{k=1}^{n-1} (2k)} = 3^n \frac{2n!}{\prod_{k=1}^n (2k)} \quad \begin{matrix} \uparrow \\ (2n-2) \end{matrix} \\
&= 3^n \frac{(2n-1)!}{2^{n-1} \left(\prod_{k=1}^{n-1} k \right)} \\
&= 3^n \frac{(2n-1)!}{2^{n-1} (n-1)!} \quad \begin{matrix} \downarrow \\ \frac{3^n (2n)!}{2^n n!} \end{matrix}
\end{aligned}$$

$$c) \prod_{k=2}^n \left(1 - \frac{1}{k}\right) = \prod_{k=2}^n \frac{k-1}{k} = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \dots \times \frac{n-2}{n-1} \times \frac{n-1}{n} = \frac{1}{n}$$

$$d) \prod_{k=2}^n \left(1 - \frac{1}{k^2}\right) = \prod_{k=2}^n \frac{k^2-1}{k^2} = \prod_{k=2}^n \frac{(k-1)(k+1)}{k^2}$$

$$= \frac{1 \times 2}{2 \times 2} \times \frac{2 \times 3}{3 \times 3} \times \frac{3 \times 4}{4 \times 4} \times \frac{4 \times 5}{5 \times 5} \times \dots \times \frac{(n-2) \times (n-1)}{(n-1)(n-1)} \times \frac{(n-1)(n+1)}{n \times n} = \frac{n+1}{2n}$$

$$= \frac{\prod_{k=2}^n (k-1) \times \prod_{k=2}^n (k+1)}{\prod_{k=2}^n k \times \prod_{k=2}^n k} = \frac{\prod_{k=1}^{n-1} k \times \prod_{k=3}^{n+1} k}{\prod_{k=2}^n k \times \prod_{k=2}^n k} = \frac{1}{n} \times \frac{n+1}{2} = \frac{n+1}{2n}$$

$$\begin{aligned}
 e) \quad \prod_{k=1}^n \sqrt{k^2 + k} &= \sqrt{\prod_{k=1}^n k^2 + k} \\
 &= \sqrt{\prod_{k=1}^n k(k+1)} \\
 &= \sqrt{\prod_{k=1}^n k \times \prod_{k=1}^n (k+1)} \\
 &= \sqrt{n! \times \prod_{k=2}^{n+1} k} \\
 &= \sqrt{n! \times (n+1)!}
 \end{aligned}$$

Exercice bonus

Soit $x \in \mathbb{R}$, simplifier :

$$\prod_{1 \leq i, j \leq n} x^{i+j}$$

$$\prod_{1 \leq i \leq j \leq n} x^{i+j}$$

$i \times j$

$i \backslash j$	1	2	3	4	
1	1	2	3	4	$i < j$
2	2	4	6	8	
3	3	6	9	12	$i > j$
4	4	8	12	16	$i = j$

Solution : martinbroussard.fr