J-77

Polynômes (Exercices)

① En raisonnant sur les degrés / coefficients :

a) Montrer qu'on ne peut trouver P tel que $P^2 = X(X^8+1)$

Supp. P tq $P^2 = X(X^8+1)$ $\qquad P \neq 0$

$\deg P = n$

$\deg P^2 = \deg(P \times P) = 2 \deg P = 2n$

$\deg(X(X^8+1)) = \deg(X^9+X) = 9$

} absurde

b) Trouver tous les P tels que $P^3 = X^2 P$

$P = 0$ ✓

$P = X$ ✓

Analyse : Soit P tq $P^3 = X^2 P$

Si $P \neq 0$, $\deg P = n$ $\quad \deg P^3 = 3n$ $\quad \deg X^2 P = n+2$

$3n = n+2 \iff n = 1$ $\quad P(X) = aX + b$ (c1)

$P^3(X) = (aX+b)^3 = \underline{a^3} X^3 + 3a^2 b X^2 + 3ab^2 X + \underbrace{b^3}_{=0}$

$(X^2 P)(X) = \underline{a} X^3 + b X^2 + 0X + 0$

$a = a^3 \iff a^3 - a = a(a^2-1) = 0 \iff a \in \{-1; 1\}$

$P(X) = aX$ (c2) $\quad a \neq 0$

$\begin{cases} P(X) = -X \\ P(X) = X \end{cases}$

Synthèse $\quad P(X) = 0$ ✓ $\quad P(X) = X$ ✓ $\quad (-X)^3 = X^2(-X)$ ✓

$P(X) = -X$ ✓

Concl : $\mathcal{S} = \{0, -X, X\}$

c) Trouver tous les P tels que $P(X^2) = (X^2+1)P(X)$ $P=0$ ✓

Analyse : Supp. P tq $P(X^2) = (X^2+1)P(X)$

Si $P \neq 0$, $\deg P = n$ $P(X^2) = \sum_{k=0}^{n} a_k (X^2)^k$ ← $\deg P(X^2) = 2n$ ⇝ $n=2$

$\deg\left((X^2+1)P(X)\right) = n+2$

$P(X) = \underset{(a\neq 0)}{aX^2+bX+c}$ (c1)

$P(X^2) = aX^4 + bX^2 + c + 0X^3 + 0X$

$\begin{cases} a+c=0 \\ b=0 \end{cases}$ $c=-a$

$(X^2+1)P(X) = aX^4 + bX^3 + (a+c)X^2 + bX + c$

$P(X) = aX^2 - a = a(X^2-1)$ (c2)

Synthèse $P(X) = a(X^2-1)$ $a \in \mathbb{R}$ Cl !

$P(X^2) = a(X^4-1) = a(X^2+1)(X^2-1) = (X^2+1)P(X)$ ✓

② a) Diviser $5x^4-7x^3+x^2-9$ par x^2+3x+6

$5x^4-7x^3+x^2-0x-9$ | x^2+3x+6

$\ominus\ 5x^4+15x^3+30x^2$ | $5x^2-22x+37$

$-22x^3-29x^2-0x-9$

$\ominus\ -22x^3-66x^2-132x$

$37x^2+132x-9$

$\ominus\ 37x^2+111x+222$

$21x-231$

$5x^4-7x^3+x^2-9$

$=$

$(x^2+3x+6)(5x^2-22x+37)+21x-231$

b) x^4 par $x^3 - 3x^2 + x - 5$

$$
\begin{array}{l|l}
x^4 + 0x^3 + 0x^2 + 0x + 0 & x^3 - 3x^2 + x - 5 \\
\ominus\ x^4 - 3x^3 + x^2 - 5x & \hline
\hline
3x^3 - x^2 + 5x + 0 & x + 3 \\
\ominus\ 3x^3 - 9x^2 + 3x - 15 & \\
\hline
8x^2 + 2x + 15 &
\end{array}
$$

$$x^4 = (x^3 - 3x^2 + x - 5)(x + 3) + (8x^2 + 2x + 15)$$

Exercice bonus

Montrer que pour tout polynôme P :

$$P(X) - X \text{ divise } P(P(X)) - X$$

Solution : martintrousard.fr