Polynômes : Racines et factorisation (Exercices)

(1) Factoriser $X^5 + 3X^4 - 22X^3 + 38X^2 - 27X + 7$ $\quad P(1) = 0$

$$P(x) = (X-1)(aX^4 + bX^3 + cX^2 + dX + e)$$
$$= \underbrace{a}_{1}X^5 + \underbrace{(b-a)}_{3}X^4 + \underbrace{(c-b)}_{-22}X^3 + \underbrace{(d-c)}_{38}X^2 + \underbrace{(e-d)}_{-27}X \underbrace{-e}_{7}$$

$a = 1$, $b = 4$, $c = -18$, $d = 20$, $e = -7$

$$\begin{cases} P(X) = (X-1)\,Q_1(X) \\ Q_1(X) = X^4 + 4X^3 - 18X^2 + 20X - 7 \end{cases}$$

$Q_1(1) = 0$

$$Q_1(x) = (X-1)(aX^3 + bX^2 + cX + d)$$
$$= \underbrace{a}_{1}X^4 + \underbrace{(b-a)}_{4}X^3 + \underbrace{(c-b)}_{-18}X^2 + \underbrace{(d-c)}_{20}X \underbrace{-d}_{-7}$$

$a = 1$; $b = 5$; $c = -13$; $d = 7$

$$\begin{cases} P(X) = (X-1)^2\,Q_2(X) \\ Q_2(X) = X^3 + 5X^2 - 13X + 7 \end{cases}$$

$$\begin{cases} P(X) = (X-1)^2\, Q_2(X) \\ Q_2(X) = X^3 + 5X^2 - 13X + 7 \end{cases}$$

$Q_2(1) = 0$

$-1, 1, 2, \ldots$

$$Q_2(X) = (X-1)(aX^2 + bX + c)$$
$$= \underset{1}{\underline{a}}X^3 + (\underset{5}{\underline{b-a}})X^2 + (\underset{-13}{\underline{c-b}})X \underset{7}{\underline{-c}}$$

$a = 1$; $b = 6$; $c = -7$

$$\begin{cases} P(X) = (X-1)^3\, Q_3(X) \\ Q_3(X) = X^2 + 6X - 7 \end{cases}$$

$\Delta = 36 + 28 = 64$ $\quad X_{1,2} = \dfrac{-6 \pm 8}{2} \rightarrow -7$ et 1

$Q_3(X) = (X+7)(X-1)$

$P(X) = (X-1)^4 (X+7)$

② Déterminer la multiplicité de 1 dans :

$$X^{2n+1} - (2n+1) X^{n+1} + (2n+1) X^{n} - 1$$

$$= X^{2n+1} - 1 - (2n+1) X^{n} (X-1)$$

$(1-q)(1+q+\dots+q^{n}) = 1-q^{n+1}$

$$= (X-1)(1+X+\dots+X^{2n}) - (2n+1) X^{n} (X-1)$$

$$= (X-1) \underbrace{\left[\underbrace{(1+X+\dots+X^{2n})}_{2n+1 \text{ termes}} - (2n+1) X^{n} \right]}_{Q(X)}$$

$Q(1) = 0$

$Q(X) = (X-1) \dots$

$$= (X-1)\left[(1-X^{n}) + (X-X^{n}) + \dots + (X^{2n}-X^{n}) \right]$$

$$= (X-1)\left[\sum_{k=0}^{2n} (X^{k}-X^{n}) \right]$$

$$(x^0-x^n)+(\underline{x^1}-x^n)+\cdots+(x^{2n}-\underline{x^n})$$

$$= (x-1)\left[\sum_{k=0}^{2n}(x^k-x^n)\right]$$

$$= (x-1)\left[\sum_{k=0}^{n-1}(x^k-x^n) + \sum_{k=n+1}^{2n}(x^k-x^n)\right]$$

$$= (x-1)\left[\sum_{k=0}^{n-1}x^k\underline{(1-x^{n-k})} + \sum_{k=n+1}^{2n}x^n\underline{(x^{k-n}-1)}\right]$$

$$= (x-1)\left[\sum_{k=0}^{n-1}x^k(1-x)(1+\cdots+x^{n-k-1}) + \sum_{k=n+1}^{2n}x^n(x-1)(1+\cdots+x^{k-n-1})\right]$$

$$= (x-1)^2\left[-\sum_{k=0}^{n-1}x^k(1+\cdots+x^{n-k-1}) + \sum_{k=n+1}^{2n}x^n(1+\cdots+x^{k-n-1})\right]$$

$$= (x-1)^2\left[-\sum_{k=0}^{n-1}x^{n-k-1}(1+\cdots+x^k) + \sum_{k=0}^{n-1}x^n(1+\cdots+x^k)\right]$$

$$= (X-1)^2 \left[- \sum_{k=0}^{n-1} X^{n-k-1} (1 + \dots + X^k) + \sum_{k=0}^{n-1} X^n (1 + \dots + X^k) \right]$$

$$= (X-1)^2 \left[\sum_{k=0}^{n-1} (1 + \dots + X^k) \underline{\left(X^n - X^{n-k-1} \right)} \right]$$

$$X^{n-k-1} (X^{k+1} - 1)$$

$$= X^{n-k-1} (X-1)(1 + X + \dots + X^k)$$

mult.

$$= (X-1)^3 \underline{\left[\sum_{k=0}^{n-1} (1 + \dots + X^k) X^{n-k-1} (1 + X + \dots + X^k) \right]}$$

$$X = 1 \implies Q_3(X) \neq 0$$

Exercice bonus

Déterminer les racines $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ de

$$x^4 - 10x^3 + 25x^2 - 36$$

en sachant que $\lambda_1 + \lambda_2 = 5$

Solutions : martintrassard.fr