

Inégalités (Exercices)

① Soit $a \in [-3; 5]$. Encadrer $\frac{a+3b}{b^2+1} = (a+3b) \times \frac{1}{b^2+1}$
 $b \in]2; 7[$

$$\underline{-3 \leq a \leq 5}$$

$$2 < b < 7$$

$$(3 > 0) \quad \underline{6 < 3b < 21}$$

$$0 < \underline{3 < a+3b < 26}$$

(caractère $\uparrow$ sur $\mathbb{R}_+$)

$$4 < b^2 < 49$$

$$5 < b^2+1 < 50$$

(caractère $\downarrow$ sur $\mathbb{R}_+^*$)

$$\frac{1}{5} > \frac{1}{b^2+1} > \frac{1}{50}$$

$$0 < \underline{\frac{1}{50} < \frac{1}{b^2+1} < \frac{1}{5}}$$

(Produit)

$$\frac{3}{50} < \frac{a+3b}{b^2+1} < \frac{26}{5}$$

$\frac{3}{50} \cdot \frac{18}{25} < \frac{11}{5}$

$$\textcircled{2} \text{ Montrer que } \forall (x, y) \in (\mathbb{R}_+^*)^2 : \frac{2}{\frac{1}{x} + \frac{1}{y}} \stackrel{\textcircled{2}}{\leq} \sqrt{xy} \stackrel{\textcircled{1}}{\leq} \frac{x+y}{2}$$

$$\textcircled{1} \quad \sqrt{xy} \leq \frac{x+y}{2} \xLeftrightarrow{\text{carré str. } \sqrt{\cdot}} xy \leq \frac{x^2 + 2xy + y^2}{4} \Leftrightarrow 0 \leq \frac{x^2 - 2xy + y^2}{4} \\ \Leftrightarrow 0 \leq \frac{(x-y)^2}{4} \quad \text{vrai}$$

$$\textcircled{2} \quad \frac{2}{\frac{1}{x} + \frac{1}{y}} = \frac{2xy}{y+x} = \frac{xy}{\left(\frac{x+y}{2}\right)} \quad \frac{x+y}{2} \stackrel{\textcircled{1}}{\geq} \sqrt{xy} \quad \downarrow \text{inv str. } \mathbb{R}_+^* \\ \frac{1}{\left(\frac{x+y}{2}\right)} \leq \frac{1}{\sqrt{xy}} \quad \downarrow \text{inv str. } \mathbb{R}_+^* \\ \frac{2}{\frac{1}{x} + \frac{1}{y}} = \frac{xy}{\left(\frac{x+y}{2}\right)} \leq \frac{xy}{\sqrt{xy}} = \sqrt{xy}$$

②

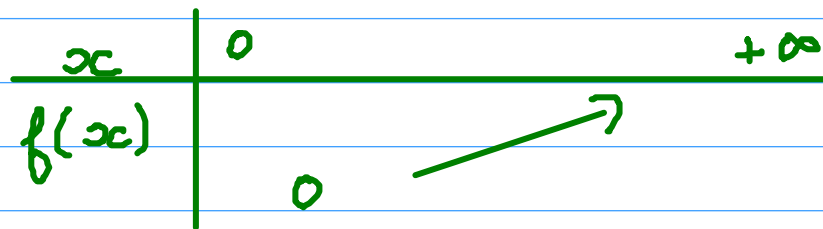
③ Montrer que $\forall x \in \mathbb{R}_+ : x - \frac{x^2}{2} \leq \ln(1+x) \leq x$

$$f: x \mapsto \ln(1+x) - x + \frac{x^2}{2}$$

$$f'(x) \stackrel{①}{=} \frac{1}{1+x} - 1 + x = \frac{x^2}{1+x}$$

≥ 0
 $\geq 1 > 0$

$$f'(x) \geq 0$$



$$f(x) \geq 0$$

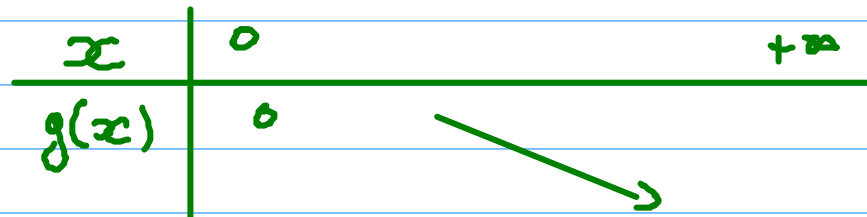
$$\ln(1+x) \geq x - \frac{x^2}{2} \quad \checkmark$$

$$g: x \mapsto \ln(1+x) - x$$

$$g'(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x}$$

≥ 0
 > 0

$$g'(x) \leq 0$$



$$g(x) \leq 0$$

$$\ln(1+x) \leq x \quad \checkmark$$

Exercice bonus

Montrer les inégalités suivantes :

$$\textcircled{1} \quad (a+b)^4 \leq 8(a^4 + b^4) \quad (a, b) \in \mathbb{R}^2$$

$$\textcircled{2} \quad a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a+b+c) \quad (a, b, c) \in \mathbb{R}^3$$

Solutions : martinroussard.fr