Valeur absolue, partie entière (Exercices)

① Soit $(x,y) \in \mathbb{R}^2$, montrer $|x| + |y| \leq \underbrace{|x+y|}_{\leq |x|+|y|} + \underbrace{|x-y|}_{\geq ||x|-|y||}$

$$a = x + y \qquad a+b = 2x \qquad x = \frac{a+b}{2}$$

$$b = x - y \qquad a-b = 2y \qquad y = \frac{a-b}{2}$$

$$\left|\frac{a+b}{2}\right| + \left|\frac{a-b}{2}\right| \leq |a| + |b| \quad ?$$

$$\left|\frac{a+b}{2}\right| = \frac{|a+b|}{2} \leq \frac{|a|+|b|}{2}$$

$$\left|\frac{a-b}{2}\right| = \frac{|a-b|}{2} = \frac{|a+(-b)|}{2} \leq \frac{|a|+|-b|}{2} = \frac{|a|+|b|}{2}$$

$\geq 0 \quad x \geq -3 \qquad \geq 0 \quad x \geq 4$

$< 0 \quad x < -3 \qquad < 0 \quad x < 4$

② Résoudre dans $\mathbb{R}$, $|x^2+2x-8| + |x+3| \leq |x-4|$

Signe de x^2+2x-8 $\quad \Delta = 4 - 4 \times 1 \times (-8) = 36 > 0 \quad (x+4)(x-2)$

$-4 \qquad 2$

$|x^2+2x-8| + |x+3| \leq |x-4|$

$-\infty$					
	x^2+2x-8	$-x-3 \leq -x+4$	$\underline{x^2+2x-15 \leq 0}$	$[-5, \cancel{3}]$ -4	
-4	0				
	$-x^2-2x+8$	$-x-3 \leq -x+4$	$-x^2-2x+1 \leq 0$	$\begin{cases} x \leq -1-\sqrt{2} \\ (x \geq -1+\sqrt{2}) \end{cases}$	$[-4, -3]$
-3		0			
	$-x^2-2x+8$	$+x+3 \leq -x+4$	$-x^2+7 \leq 0$	$\begin{cases} x \leq -\sqrt{7} \\ (x \geq \sqrt{7}) \end{cases}$	$[-3; -\sqrt{7}]$
2	0				
	x^2+2x-8	$+x+3 \leq -x+4$	$x^2+4x-9 \leq 0$	$[-2-\sqrt{13}, -2+\sqrt{13}]$	X
4		0			
	x^2+2x-8	$+x+3 \leq x-4$	$x^2+2x-1 \leq 0$	$[1-\sqrt{2}; 1+\sqrt{2}]$	X
$+\infty$					

$S = [-5; -\sqrt{7}]$

③ Soit $n \in \mathbb{N}$, simplifier $\displaystyle\sum_{k=1}^{n^2} \lfloor \sqrt{k} \rfloor = S_n$

$n=1$ $\quad \lfloor \sqrt{1} \rfloor = 1$

$n=2$ $\quad \lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \lfloor \sqrt{4} \rfloor = 5$

$n=3$ $\quad S_2 + \lfloor \sqrt{5} \rfloor + \lfloor \sqrt{6} \rfloor + \lfloor \sqrt{7} \rfloor + \lfloor \sqrt{8} \rfloor + \lfloor \sqrt{9} \rfloor = 16$

$n=4$ $\quad S_3 + \underbrace{\lfloor \sqrt{10} \rfloor + \dots + \lfloor \sqrt{15} \rfloor}_{6 \times 3} + \underbrace{\lfloor \sqrt{16} \rfloor}_{4} = 38$ $\quad 4 \times 2$

$n=5$ $\quad S_4 + \underbrace{\lfloor \sqrt{17} \rfloor + \dots + \lfloor \sqrt{24} \rfloor}_{8 \times 4} + \underbrace{\lfloor \sqrt{25} \rfloor}_{5}$

$+4$

$+11$

$+22$

$$S_{n+1} - S_n = \sum_{k=1}^{(n+1)^2} \lfloor \sqrt{k} \rfloor - \sum_{k=1}^{n^2} \lfloor \sqrt{k} \rfloor = \sum_{k=n^2+1}^{(n+1)^2} \lfloor \sqrt{k} \rfloor = \overbrace{\sum_{k=n^2+1}^{(n+1)^2-1}}^{n^2+2n} \underbrace{\lfloor \sqrt{k} \rfloor}_{=n} + (n+1)$$

$n^2 \leq k < (n+1)^2 \implies n \leq \sqrt{k} < n+1 \implies \lfloor \sqrt{k} \rfloor = n$

$$= 2n \times n + (n+1) = 2n^2 + n + 1$$

$$S_{n+1} - S_n = 2n^2 + n + 1$$

$$\cancel{S_2} - S_1 + \cancel{S_3} - \cancel{S_2} + \cdots + S_n - \cancel{S_{n-1}}$$

$$S_n - \underbrace{S_1}_{=1} = \sum_{k=1}^{n-1} \left(S_{k+1} - S_k\right) = \sum_{k=1}^{n-1} \left(2k^2 + k + 1\right)$$

$$= 2\sum_{k=1}^{n-1} k^2 + \sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1} 1$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= 2\,\frac{(n-1)n(2n-1)}{6} + \frac{(n-1)n}{2} + (n-1)$$

$$S_n = \frac{(n-1)n(2n-1)}{3} + \frac{(n-1)n}{2} + (n-1) + 1$$

$$S_n = \frac{n(4n^2 - 3n + 5)}{6}$$

Exercice bonus

Montrer que $\forall n \in \mathbb{N}^*$, $\forall x \in \mathbb{R}$,

$$\left\lfloor \frac{\lfloor nx \rfloor}{n} \right\rfloor = \lfloor x \rfloor$$

$$x-1 < \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$

Solutions : martintrouvard.fr