Coefficients binomiaux

$n \in \mathbb{N},\ k \in \mathbb{N}$

Coefficient binomial

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

"k parmi n"

Interprétation combinatoire

k choix parmi n éléments

→ sans remise

1^{er} : n possibilités

2^{e} : $n-1$ "

$\vdots$

$k^{ème}$: $n-k+1$ "

$n \times (n-1) \times \dots \times (n-k+1)$

$$\prod_{i=n-k+1}^{n} i = \frac{\prod_{i=1}^{n} i}{\prod_{i=1}^{n-k} i} = \frac{n!}{(n-k)!}$$

→ sans ordre

$(1, \dots, 10)$

3 ?

$(1,2,3)$

$(1,3,2)$

$(2,1,3)$ ...

$k!$ permutations

$$\frac{n!}{(n-k)!} \times \frac{1}{k!}$$

Règles de calcul

$$\frac{n!}{k!(n-k)!}$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{n}{1} = \binom{n}{n-1} = n$$

$$k > n, \quad \binom{n}{k} = 0$$

$$0 \leq k \leq n, \quad \binom{n}{k} = \binom{n}{n-k}$$

choisir k $\Leftrightarrow$ ne pas choisir $n-k$

Pascal $$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

$\binom{n}{k}$ (lignes n, colonnes k)	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Triangle de Pascal

Binôme de Newton

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \qquad a^{n-k} b^k$$

Preuve

Récurrence

Hér : $(a+b)^{n+1} = (a+b)\left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\right)$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1}$$

Pascal

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

$$= \sum_{k=1}^{n+1} \binom{n}{k-1} a^k b^{n-(k-1)} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1}$$

$n-k+1$

$k=n+1 \longrightarrow a^{n+1}$ $\qquad k=0 \longrightarrow b^{n+1}$

$$= a^{n+1} + \sum_{k=1}^{n} \left[\underbrace{\binom{n}{k-1} + \binom{n}{k}}_{\binom{n+1}{k}}\right] a^k b^{(n+1)-k} + b^{n+1}$$

$$\binom{n+1}{n+1} a^{n+1} b^{(n+1)-(n+1)} \qquad \binom{n+1}{0} b^{n+1-0} a^0$$

Binôme avec $b = 1$ $$(a+1)^n = \sum_{k=0}^{n} \binom{n}{k} a^k$$

Exemple $$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

Binôme "dérivé" $$f : x \mapsto (x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$f'(x) = n(x+1)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k\, x^{k-1}$$

$$x = 1 \implies n\, 2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k$$

$$x = -1 \implies 0 = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k$$

À vous de jouer !

① Soit $(n,p) \in \mathbb{N}^2$ tel que $n \geq p$. Montrer que :

$$\sum_{k=p}^{n} \binom{k}{p} = \binom{n+1}{p+1}$$

Triangle ?

② Simplifier les sommes :

$$\sum_{k=i}^{n} \binom{n-i}{n-k} \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \qquad \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k}$$