# Coefficients binomiaux (Exercices)

① Soit $(n, p) \in \mathbb{N}^2$ tel que $n \geqslant p$. Montrer que :

$$\sum_{k=p}^{n} \binom{k}{p} = \binom{n+1}{p+1}$$

$n = 3$

$$\binom{4}{p+1}$$

$\rightarrow p = 0$

$$\binom{0}{0}+\binom{1}{0}+\binom{2}{0}+\binom{3}{0}=\binom{4}{1}$$

$\rightarrow p = 1$

$$\binom{1}{1}+\binom{2}{1}+\binom{3}{1}=\binom{4}{2}$$

$\rightarrow p = 2$

$$\binom{2}{2}+\binom{3}{2}=\binom{4}{3}$$

$\rightarrow p = 3$

$$\binom{3}{3}=\binom{4}{4}$$

| $\binom{n}{k}$ | $k$: 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $n$: 0 | 1 | | | | | | |
| 1 | 1 | 1 | | | | | |
| 2 | 1 | 2 | 1 | | | | |
| 3 | 1 | 3 | 3 | 1 | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |

$P(n)$ : " $\forall p \leq n, \quad \sum_{k=p}^{n} \binom{k}{p} = \binom{n+1}{p+1}$ "

**Init** : $\sum_{k=0}^{0} \binom{k}{p} = \binom{0}{0} = 1 = \binom{0+1}{0+1}$ ✓

**Héré** : Soit $n \in \mathbb{N}$ tq $P(n)$ vraie

Soit $p \leq n+1$ — $p \leq n$ ✓ ; $p = n+1$

Si $p \leq n$, $\sum_{k=p}^{n+1} \binom{k}{p} = \sum_{k=p}^{n} \binom{k}{p} + \binom{n+1}{p}$

$$= \binom{n+1}{p+1} + \binom{n+1}{p} = \binom{n+2}{p+1} = \binom{(n+1)+1}{p+1} \checkmark$$

Si $p = n+1$ $\quad \sum_{k=p}^{n+1} \binom{k}{p} = \binom{n+1}{n+1} = 1 = \binom{n+2}{n+2} = \binom{(n+1)+1}{p+1}$ ✓

② Simplifier les sommes :

$$\sum_{k=i}^{n} \binom{n-i}{n-k} = \binom{n-i}{n-i} + \binom{n-i}{n-(i+1)} + \dots + \binom{n-i}{0}$$

$$= \sum_{k=0}^{n-i} \binom{n-i}{k} 1^{k} 1^{(n-i)-k} = 2^{n-i}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \times 1^{n-k} = (-1+1)^n = 0$$

$+1$ si $k$ pair
$-1$ sinon

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$$

$$f : x \longmapsto (1+x)^{n} = \sum_{k=0}^{n} \binom{n}{k} x^{k}$$

Primitive

$$G(x) = \frac{(1+x)^{n+1}}{n+1} \qquad F(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+1}}{k+1}$$

$$G(1) = \frac{2^{n+1}}{n+1} \qquad F(1) = \sum_{k=0}^{n} \binom{n}{k} \times \frac{1}{k+1}$$

$$G(0) = \frac{1}{n+1} \qquad F(0) = 0$$

$$G(1) - G(0) = F(1) - F(0)$$

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{2^{n+1} - 1}{n+1}$$

## Exercice bonus

Montrer que pour $(n, p, q) \in \mathbb{N}^3$ avec $n \leq p$ et $n \leq q$ :

$$\sum_{k=0}^{n} \binom{p}{k}\binom{q}{n-k} = \binom{p+q}{n}$$

Triangle ?
(n fixé)

Solutions : martinbrossard.fr