Trigonométrie : Produits, sommes, équations (Exercices)

$\rightarrow \cos(a+b) = \cos a \cos b - \sin a \sin b$

$\rightarrow \cos(a-b) = \cos a \cos b + \sin a \sin b$

$\sin(a+b) = \cos a \sin b + \sin a \cos b$

$\sin(a-b) = \sin a \cos b - \cos a \sin b$

① Résoudre dans $\mathbb{R}$ les équations :

a) $\cos x + \cos(4x) = 0$

$$\begin{cases} a+b = 4x \\ a-b = x \end{cases} \iff \begin{cases} a = \dfrac{5x}{2} \\ b = \dfrac{3x}{2} \end{cases}$$

$$\iff 2\cos\left(\frac{5x}{2}\right)\cos\left(\frac{3x}{2}\right) = 0$$

$$\iff \cos\left(\frac{5x}{2}\right) = 0 \text{ ou } \cos\left(\frac{3x}{2}\right) = 0$$

$$\iff \frac{5x}{2} \equiv \frac{\pi}{2}\ [\pi] \text{ ou } \frac{3x}{2} \equiv \frac{\pi}{2}\ [\pi]$$

$$\frac{5x}{2} = \frac{\pi}{2} + k\pi \iff 5x = \pi + 2k\pi \iff x = \frac{\pi}{5} + k \times \frac{2\pi}{5}$$

$$\iff x \equiv \frac{\pi}{5}\ \left[\frac{2\pi}{5}\right] \text{ ou } x \equiv \frac{\pi}{3}\ \left[\frac{2\pi}{3}\right]$$

$$\cos^2(x) - \sin^2(x) = 1 - 2\sin^2 x$$

b) $\cos(2x) + \sin(3x) = 0$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\cos(a-b) = \cos a \cos b + \sin a \sin b$

$\sin(a+b) = \cos a \sin b + \sin a \cos b$

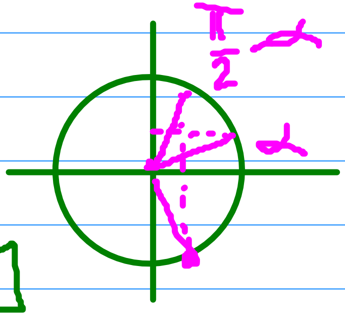
$\sin(a-b) = \sin a \cos b - \cos a \sin b$

$$\cos x \underbrace{\sin 2x}_{2\cos x \sin x} + \sin x \underbrace{\cos 2x}_{\cos^2 x - \sin^2 x} = 3 \underbrace{\cos^2 x}_{(1-\sin^2 x)} \sin x - \sin^3 x$$

X $\iff -4\sin^3 x - 2\sin^2 x + 3\sin x + 1 = 0$ $\quad X = \sin x$

$$\iff \cos(2x) + \cos\left(\frac{\pi}{2} - 3x\right) = 0$$

$$\iff 2\underbrace{\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}_{=0} \underbrace{\cos\left(\frac{5x}{2} - \frac{\pi}{4}\right)}_{=0} = 0$$



$\sin(\alpha) = \cos\left(\frac{\pi}{2} - \alpha\right)$

$$\iff \frac{\pi}{4} - \frac{x}{2} = \frac{\pi}{2}\ [\pi] \quad \text{ou} \quad \frac{5x}{2} - \frac{\pi}{4} = \frac{\pi}{2}\ [\pi]$$

↓ (détail)

$$\iff x \equiv -\frac{\pi}{2}\ [2\pi] \quad \text{ou} \quad x \equiv \frac{3\pi}{10}\ \left[\frac{2\pi}{5}\right]$$

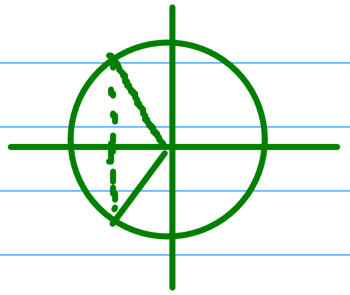
$$3\cos^2 x \sin x - \sin^3 x$$

c) $\sin(x) + \underbrace{\sin(2x)}_{2\cos x \sin x} + \sin(3x) = 0$

$$\iff \sin x \left(\cancel{1} + 2\cos x + 3\cos^2 x \underbrace{- \cancel{\sin^2 x}}_{+\cos^2 x} \right) = 0$$

$$\iff \sin x \left(2\cos x + 4\cos^2 x \right) = 0$$

$$\iff \underbrace{\sin x}_{=0} \text{ ou } \underbrace{\cos x}_{=0} \times 2 \text{ ou } \underbrace{(1 + 2\cos x)}_{=0} = 0$$



$$1 + 2\cos x = 0$$

$$\iff x \equiv 0\ [\pi] \quad \text{ou} \quad x \equiv \frac{\pi}{2}\ [\pi] \quad \text{ou} \quad \cos x = -\frac{1}{2}$$

$$x \equiv \frac{2\pi}{3}\ [2\pi] \text{ ou } x \equiv -\frac{2\pi}{3}\ [2\pi]$$

$$\oplus \begin{cases} \sin(a+b) = \cancel{\cos a \sin b} + \sin a \cos b \\ \sin(a-b) = -\cancel{\cos a \sin b} + \sin a \cos b \end{cases} \quad 2$$

$$\sin x + \sin 3x = 2\sin(2x)\cos x$$

② Déterminer la valeur maximale (éventuelle) atteinte par :

$$f : x \longmapsto \underbrace{3}_{a}\cos x \underbrace{- 2}_{b} \sin x$$

$$\sqrt{a^2+b^2} = \sqrt{9+4} = \sqrt{13}$$

$$\forall x \in \mathbb{R},\ f(x) = \sqrt{13}\left(\overbrace{\frac{3}{\sqrt{13}}}^{u} \cos x \overbrace{- \frac{2}{\sqrt{13}}}^{v} \sin x \right)$$

$u^2+v^2 = 1 \qquad \exists \varphi \in \mathbb{R} : \begin{cases} u = \cos\varphi \\ v = \sin\varphi \end{cases}$

$$f(x) = \sqrt{13}\left(\cos\varphi \cos x + \sin\varphi \sin x \right) = \sqrt{13} \cos(x-\varphi)$$

$\underline{f(x) \leq \sqrt{13}} \qquad \underline{f(\varphi) = \sqrt{13}\cos(0) = \sqrt{13}} \qquad \max = \sqrt{13}$

Exercice bonus

Soit α, β, γ les mesures des angles d'un triangle.

Montrer que :

$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = -4 \cos\frac{3\alpha}{2} \cos\frac{3\beta}{2} \cos\frac{3\gamma}{2}$$

Solutions : martinrausard.fr