Suites récurrentes

Suite récurrente d'ordre 1

Soit $E \subset \mathbb{R}$ et $f : E \to E$

$$u_{n+1} = f(u_n)$$

$$u_{n+1} = \sqrt{u_n - 3}$$

$$u_{n+1} = 3u_n^2 - u_n$$

Monotonie de (u_n) : Si f est croissante

$f : x \longmapsto x^2$

$u_0 \in \mathbb{R}_+$

$u_1 = f(u_0)$

$u_2 = f(u_1)$

$\vdots$

$u_0 = 2, u_1 = 4, u_2 = 16, \dots, u_n = 2^{2^n}$ $\nearrow$

$u_0 = \frac{1}{2}, u_1 = \frac{1}{4}, u_2 = \frac{1}{16}, \dots, u_n = 2^{-2^n}$ $\searrow$

⚠ $u_n = f(n)$ a la m̂ monotonie que f

Init : $u_0 \geq u_1$?

Non / oui

$u_{n+1} = f(u_n)$ $\quad$ (u_n) monotone

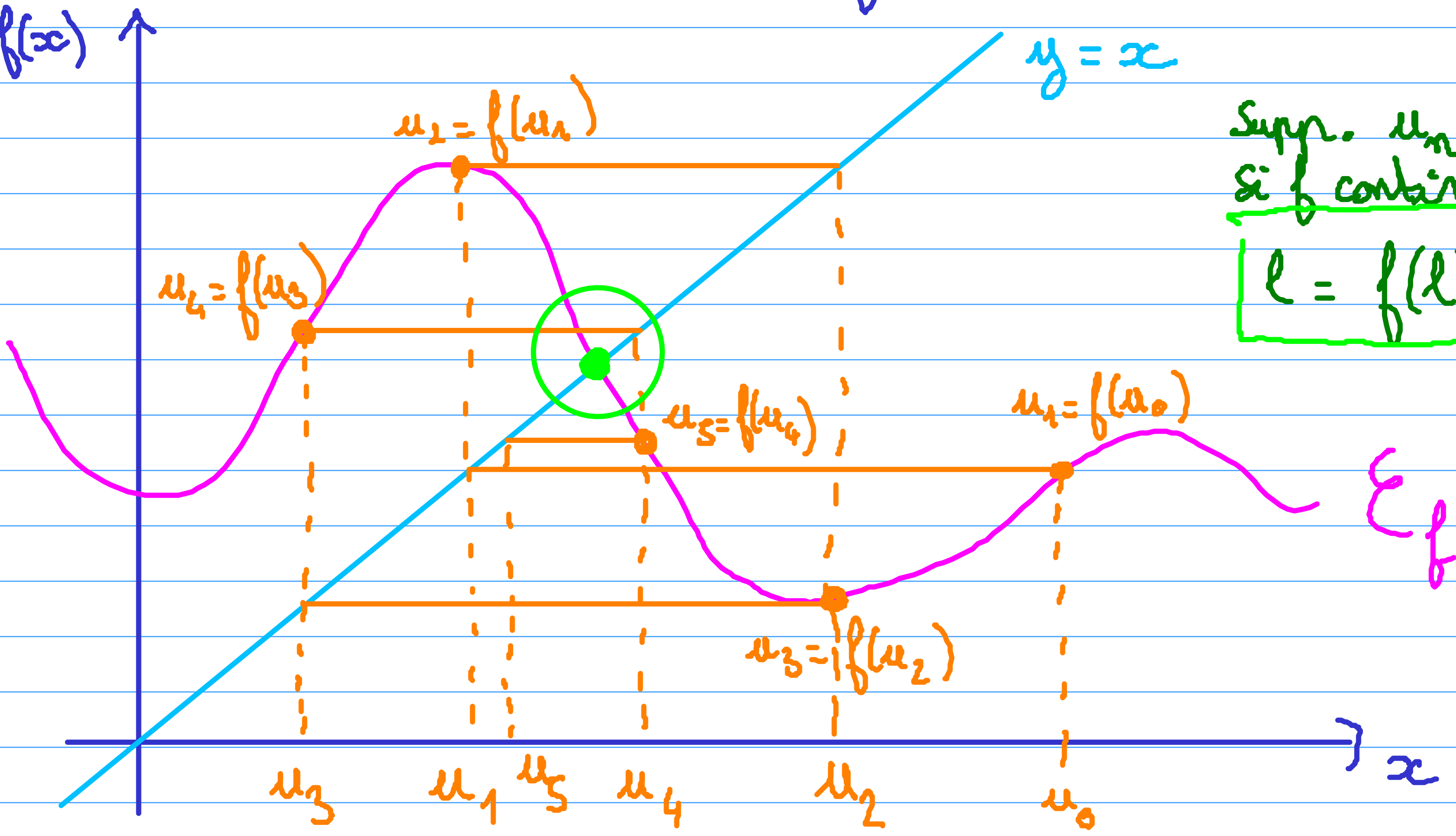
$$u_n \leq u_{n+1} \xrightarrow{f \nearrow} u_{n+1} \leq u_{n+2}$$

$$u_n \geq u_{n+1} \xrightarrow{f \nearrow} u_{n+1} \geq u_{n+2}$$

Construction des termes

$$u_{n+1} = f(u_n)$$

$$\begin{cases} u_0 = 2 \\ u_{n+1} = u_n^2 \end{cases}$$ ~~cv~~



Supp. $u_n \to \ell$

si f continue

$$\boxed{\ell = f(\ell)}$$

Suites arithmético-géométriques

$$u_{n+1} = f(u_n) = a u_n + b$$

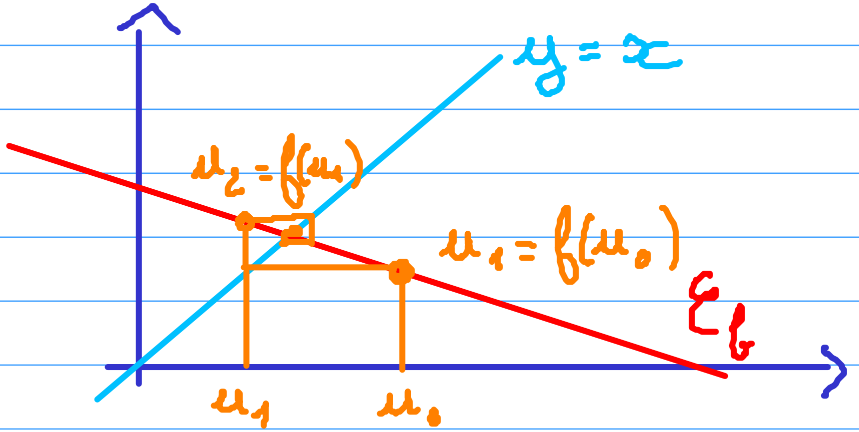
$a \neq 0$ cst. $\quad a \neq 1$ arith. $\quad b \neq 0$ géom.

$$v_n = u_n - \underset{?}{\ell} = u_n - \frac{b}{1-a}$$

$$v_{n+1} = u_{n+1} - \ell = a u_n + b - \ell = a(v_n + \ell) + b - \ell$$

$$= a v_n + \underbrace{(a\ell + b - \ell)}_{=0}$$

$$a\ell + b - \ell = 0 \iff \ell = \frac{b}{1-a}$$



$$v_n = v_0\, a^n = \left(u_0 - \frac{b}{1-a}\right) a^n$$

$$u_n = \left(u_0 - \frac{b}{1-a}\right) a^n + \frac{b}{1-a}$$

Si $-1 < a < 1$ $\quad u_n \longrightarrow \frac{b}{1-a}$ $\quad$ sinon pas de lim.

Suites récurrentes linéaires d'ordre 2

$$\begin{cases} u_0 \in \mathbb{R}, u_1 \in \mathbb{R} \\ u_{n+2} = a u_{n+1} + b u_n \end{cases}$$

$b \neq 0$

Équation caractéristique : $u_n = x^n$

$$x^{n+2} = a x^{n+1} + b x^n$$
$$x^2 = a x + b$$

λ solution de $x^2 = ax + b$ $\qquad u_n = \lambda^n$! incomplet

$$u_{n+2} = \lambda^{n+2} = \lambda^n \lambda^2 = \lambda^n (a\lambda + b) = a\lambda^{n+1} + b\lambda^n = a u_{n+1} + b u_n$$

⚠ $u_0 = u_1 = 1$ $\qquad u_{n+2} = 3u_{n+1} - 2u_n$ $\qquad x^2 = 3x - 2 \longrightarrow \lambda = 1$ ou $\cancel{\lambda = 2}$

2 solutions (x_1, x_2) $\qquad u_n = \lambda x_1^n + \mu x_2^n$ $\qquad \begin{cases} u_0 = \lambda + \mu \\ u_1 = \lambda x_1 + \mu x_2 \end{cases}$

1 solution (x_0) $\qquad u_n = \lambda x_0^n + \mu n x_0^n$ $\qquad \begin{cases} u_0 = \lambda \\ u_1 = \lambda x_0 + \mu x_0 \end{cases}$

Pas de solution $\longrightarrow$ Complexes

$$\begin{cases} u_0 \in \mathbb{R},\ u_1 \in \mathbb{R} \\ u_{n+2} = a u_{n+1} + b u_n \end{cases}$$

$x^2 = ax + b$

↳ Pas de solution ($\mathbb{R}$)

2 solutions complexes conjuguées $r_1 = r e^{i\theta}$ $r_2 = r e^{-i\theta}$ $(\overline{r_1})$

$$u_n = \underbrace{\lambda\ r_1^n + \mu\ r_2^n}_{\in \mathbb{C}} = \lambda r^n e^{in\theta} + \mu r^n e^{-in\theta}$$

$$\begin{cases} \lambda + \mu = u_0 \\ \lambda r_1 + \mu \overline{r_1} = u_1 \end{cases} \qquad \lambda = \frac{u_1 - \overline{r_1} u_0}{r_1 - \overline{r_1}} \qquad \mu = \frac{u_1 - r_1 u_0}{\overline{r_1} - r_1} = \overline{\lambda}$$

$$u_n = \lambda r^n e^{in\theta} + \overline{\lambda r^n e^{in\theta}} = 2\,\mathrm{Re}(\underbrace{\lambda}_{\in\mathbb{C}}\ \underbrace{r^n}_{\in\mathbb{R}}\ \underbrace{e^{in\theta}}_{\in\mathbb{C}})$$

$\mathrm{Re}(zz') = \mathrm{Re}\,z\,\mathrm{Re}\,z' - \mathrm{Im}\,z\,\mathrm{Im}\,z'$

$$= 2r^n \left(\mathrm{Re}(\lambda) \cos(n\theta) - \mathrm{Im}(\lambda) \sin(n\theta) \right)$$

À vous de jouer !

① Étudier la suite $\begin{cases} u_0 = 1 \\ u_{n+1} = \sqrt{u_n + 6} \end{cases}$

→ bornes ? → variations ? → limite ?

② Donner le terme général des suites :

a) $\begin{cases} u_0 = u_1 = 1 \\ u_{n+2} = u_{n+1} + u_n, \quad \forall n \in \mathbb{N} \end{cases}$

b) $\begin{cases} u_0 = 1, \quad u_1 = 2 \\ u_{n+2} = u_{n+1} - u_n, \quad \forall n \in \mathbb{N} \end{cases}$

① Eq. car.

② Formule selon le cas