Suites récurrentes (Exercices)

① Étudier la suite $\begin{cases} u_0 = 1 \\ u_{n+1} = \sqrt{u_n + 6} \end{cases}$

$u_1 = \sqrt{7} \simeq 2,6$

$u_2 = \sqrt{8,6} \simeq 2,9$ $\nearrow$

$u_3 = \sqrt{8,9} \simeq 2,99$

$u_4 = \sqrt{8,99}$

Bornes $u_n \leq 3$? ✓

Réc. immédiate $u_n \leq 3 \Rightarrow u_n + 6 \leq 9 \xRightarrow{\nearrow \text{ de } \sqrt{\ }} u_{n+1} \leq 3$

Variations $u_n \nearrow$? ✓ $\forall n \in \mathbb{N}, u_{n+1} \geq u_n$

Réc immédiate $u_{n+1} \geq u_n \Rightarrow u_{n+1} + 6 \geq u_n + 6 \xRightarrow{\nearrow \text{ de } \sqrt{\ }} u_{n+2} \geq u_{n+1}$

Limite $u_n \nearrow$ majorée par 3 $u_n \to \ell \in \mathbb{R}$

$f : x \mapsto \sqrt{x+6}$ continue $f(u_n) \to f(\ell)$

$u_{n+1} \to \ell$

$u_n \geq u_0$

$\ell \geq u_0 = 1$

$f(\ell) = \ell \Leftrightarrow \sqrt{\ell + 6} = \ell \Leftrightarrow \ell + 6 = \ell^2 \Leftrightarrow \ell \in \{\cancel{-2}, 3\}$ $\boxed{\ell = 3}$

② a) Donner le terme général de $\begin{cases} u_0 = 0,\ u_1 = 1 \\ u_{n+2} = u_{n+1} + u_n, \quad \forall n \in \mathbb{N} \end{cases}$

Éq. car. $x^2 = x + 1 \iff x^2 - x - 1 = 0 \qquad \Delta = (-1)^2 - 4 \times 1 \times (-1) = 5$

$x_1 = \dfrac{1-\sqrt{5}}{2} \qquad x_2 = \dfrac{1+\sqrt{5}}{2}$ 2 solutions

$u_n = \lambda x_1^n + \mu x_2^n \qquad x_1 - x_2 = -\sqrt{5}$

$\begin{cases} \lambda + \mu = u_0 = 0 \\ \lambda x_1 + \mu x_2 = u_1 = 1 \end{cases} \qquad \begin{cases} \mu = -\lambda \\ \lambda(x_1 - x_2) = 1 \end{cases} \iff \begin{cases} \mu = \dfrac{1}{\sqrt{5}} \\ \lambda = -\dfrac{1}{\sqrt{5}} \end{cases}$

$$u_n = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$\frac{1}{i} = -i \qquad \frac{1}{-i} = i$

b) $\begin{cases} u_0 = 1\ ,\ u_1 = 2 \\ u_{n+2} = u_{n+1} - u_n\ ,\ \forall n \in \mathbb{N} \end{cases}$

$\Delta = (\sqrt{3}\, i)^2$

Eq. car. $x^2 = x - 1 \iff x^2 - x + 1 = 0$

$\Delta = (-1)^2 - 4 = -3 < 0$

$r_1 = \frac{1 - \sqrt{3}\, i}{2} \qquad r_2 = \frac{1 + \sqrt{3}\, i}{2} \qquad u_n = \underbrace{\lambda}_{\in \mathbb{C}} r_1^n + \underbrace{\mu}_{\in \mathbb{C}} r_2^n$

$\overline{r_1} =$

$r_1 - r_2 = -\sqrt{3}\, i$

$\begin{cases} \lambda + \mu = u_0 = 1 \\ \lambda r_1 + \mu r_2 = u_1 = 2 \end{cases} \iff \begin{cases} \lambda r_1 + \mu r_1 = r_1 \\ \lambda r_1 + \mu r_2 = 2 \end{cases} \implies \mu = \frac{r_1 - 2}{r_1 - r_2} = \frac{-\frac{3}{2} - \frac{\sqrt{3}}{2} i}{-\sqrt{3}\, i}$

$\lambda + \mu = 1 \qquad \lambda = \frac{1}{2} + \frac{\sqrt{3}}{2} i \qquad \lambda = \overline{\mu} \qquad \mu = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

$r_2 = e^{i\frac{\pi}{3}}$

$r_1 = e^{-i\frac{\pi}{3}}$

$u_n = \overline{\mu r_2^n} + \mu r_2^n = 2\,\mathrm{Re}(\mu r_2^n)$

λr_1^n

$= 2\left(\mathrm{Re}(\mu) \cos\left(\frac{n\pi}{3}\right) - \mathrm{Im}(\mu) \sin\left(\frac{n\pi}{3}\right)\right)$

$= \cos\left(\frac{n\pi}{3}\right) + \sqrt{3}\, \sin\left(\frac{n\pi}{3}\right)$

Exercice bonus

Soit $(a, b) \in \mathbb{R}^2$ 1

Donner l'expression du terme général de la suite :

$$\begin{cases} u_0 \in \mathbb{C} \\ u_{n+1} = a u_n + b \overline{u_n} \end{cases}$$

Pourquoi est-ce plus compliqué si $(a, b) \in \mathbb{C}^2$?

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Solutions : martintroussard.fr