Injectivité, surjectivité, bijectivité (Exercices)

① Les applications suivantes sont-elles injectives? surjectives? bijectives?
Si non, restreindre l'ens. de départ / arrivée pour qu'elles le soient.

a) $f_1 : \begin{cases} \mathbb{N} \longrightarrow \mathbb{N} \\ n \longmapsto n+1 \end{cases}$

<u>Inj</u> $f(a) = f(b) \Rightarrow a+1 = b+1 \Rightarrow a = b$ $\quad$ f injective

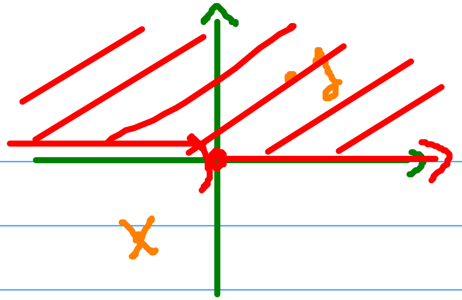
<u>Surj</u> 0 n'a pas d'antécédent $\quad$ $f : \begin{cases} \mathbb{N} \longrightarrow \mathbb{N}^* \\ n \longmapsto n+1 \end{cases}$ bijective

b) $f_2 : \begin{cases} \mathbb{Z} \longrightarrow \mathbb{Z} \\ n \longmapsto n+1 \end{cases}$ $\quad$ <u>Inj</u> : idem

<u>Surj</u> : Soit $y \in \mathbb{Z}$ $\quad$ On pose $x = y-1 \in \mathbb{Z}$ et $f(x) = y$ ✓

bijective

c) $f_3 : \begin{cases} \mathbb{C} \longrightarrow \mathbb{C} \\ z \longmapsto z^2 \end{cases}$



Inj : $f(1) = f(-1)$

$f(i) = f(-i)$

$z \in \mathbb{C}$ $\quad \exists\, r \in \mathbb{R}_+ , \theta \in]-\pi, \pi] : z = re^{i\theta}$ $\quad f(z) = r^2 e^{2i\theta}$

$r \in \mathbb{R}_+ \longmapsto r^2$ injective $\quad r_1^2 = r_2^2 \quad r_1 = \pm r_2 \quad \begin{matrix} r_1 \geq 0 \\ r_2 \geq 0 \end{matrix} \quad r_1 = r_2$

$\theta \in]-\pi, \pi] \longmapsto e^{i\theta}$ injective

$\theta \in]-\pi, \pi] \longmapsto e^{i2\theta}$ ✗

$e^{i2\theta'} = e^{i2\theta} \iff 2\theta' \equiv 2\theta\ [2\pi]$

$\iff \theta' \equiv \theta\ (\pi)$ Inj.

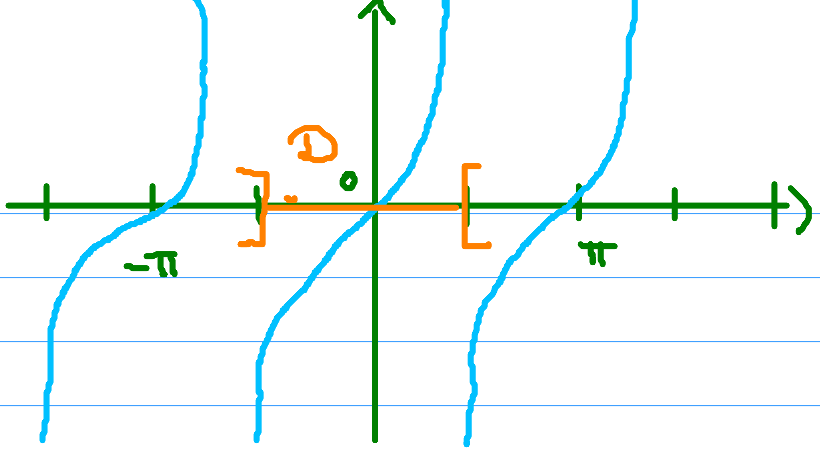
$\mathcal{D} = \{ re^{i\theta} \mid r \in \mathbb{R}_+ , \theta \in [0, \pi[\}$ $\quad \boxed{z \in \mathcal{D} \longmapsto z^2 \text{ injective}}$

Surj : $y \in \mathbb{C}$ $\quad y = re^{i\theta}$ où $r \in \mathbb{R}_+$, $\theta \in [0, 2\pi[$ $\quad \frac{\theta}{2} \in [0, \pi[$

On pose $z = \sqrt{r}\, e^{i\frac{\theta}{2}} \in \mathcal{D}$

$f(z) = z^2 = re^{i\theta} = y$ ✓

d) $f_4 : \begin{cases} \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\} \longrightarrow \mathbb{R} \\ x \longmapsto \tan x \end{cases}$



<u>Inj</u> : $f(0) = f(\pi)$

$D = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ $\quad \forall x \in D,\ \tan'(x) = \frac{1}{\cos^2 x} > 0$ $\quad$ tan str $\nearrow$, injective

<u>Surj</u> : $\lim\limits_{x \to \frac{\pi}{2}^{\ominus}} \tan x = \lim\limits_{x \to \frac{\pi}{2}^-} \frac{\sin x \,)\, 1}{\cos x \,)\, 0^+} = +\infty$ $\qquad \lim\limits_{x \to -\frac{\pi}{2}^{\oplus}} \tan x = -\infty$

TVi : $\forall y \in \mathbb{R}$, si $y > 0$ $\qquad \exists \eta > 0 : |x - \frac{\pi}{2}| < \eta \Rightarrow f(x) > y$

$f(x_0) > y$ $\qquad y \in [f(x_1), f(x_0)]$

$\exists \eta' > 0 : |x + \frac{\pi}{2}| < \eta' \Rightarrow f(x) < -y$ $\qquad$ TVi : $\exists c \in [x_1, x_0] :$

$f(x_1) < -y$ $\qquad f(c) = y$ ✓

$$\begin{tikzcd} x \in E \arrow[r, bend left, "f"] \arrow[rr, bend right, "g \circ f"'] & f(x) \in F \arrow[r, bend left, "g"] & g\circ f(x) \in G \end{tikzcd}$$

2 Soit $f : E \to F$ et $g : F \to G$

Montrer que : a) $g\circ f$ surjective $\Rightarrow$ g surjective

Soit $y \in G$

$g\circ f$ surjective donc $\exists x \in E : g\circ f(x) = y = g(\underbrace{f(x)}_{\in F})$

b) $g\circ f$ injective $\Rightarrow$ f injective

$a, b \in E$ $\qquad f(a) = f(b)$

$$g(f(a)) = g(f(b))$$

$$g\circ f(a) = g\circ f(b)$$

$$a = b$$

$\Leftarrow$ $g\circ f$ injective

Exercice bonus

Soit $f : E \longrightarrow F$. Montrer que :

1) $\left(\forall A \subset E,\ f^{-1}(f(A)) = A\right) \Longleftrightarrow f$ injective

2) $\left(\forall B \subset F,\ f(f^{-1}(B)) = B\right) \Longleftrightarrow f$ surjective

Solutions : martintroussard.fr