Calcul d'intégrales (Exercices)

$$\int u'v = [uv] - \int uv'$$

① Calculer :

a) $\displaystyle\int_1^e \underbrace{x^2}_{u'} \underbrace{\ln x}_{v}\, dx = \left[\frac{x^3 \ln x}{3}\right]_1^e - \int_1^e \frac{x^2}{3}\, dx$

$u = \dfrac{x^3}{3}$ $v' = \dfrac{1}{x}$

$$= \frac{e^3}{3} - \left[\frac{x^3}{9}\right]_1^e = \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9} = \frac{2e^3+1}{9}$$

b) $\displaystyle\int_1^e \ln^2(x)\, dx = \left[(x\ln x - x)\ln x\right]_1^e - \int_1^e (\ln x - 1)\, dx$

$u' = \ln x$ $v = \ln x$

$u = x\ln x - x$ $v' = \dfrac{1}{x}$

$$= 0 - [x\ln x - x]_1^e + [x]_1^e$$

$$= -1 + e - 1$$

$$= e - 2$$

$$\arctan'(x) = \frac{1}{1+x^2}$$

② Calculer :

a) $\displaystyle\int_0^3 \frac{1}{x^2+9}\,dx = \frac{1}{9}\int_0^3 \frac{1}{\left(\frac{x}{3}\right)^2+1}\,dx$

$\arctan\left(\frac{x}{3}\right) \xrightarrow{\text{dér.}} \frac{1}{3} \times \frac{1}{1+\left(\frac{x}{3}\right)^2}$

$$= \frac{1}{3} \times \int_0^3 \frac{1}{3} \times \frac{1}{\left(\frac{x}{3}\right)^2+1}\,dx$$

$$= \frac{1}{3}\left[\arctan\left(\frac{x}{3}\right)\right]_0^3$$

$$= \frac{1}{3}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{12}$$

b) $\int_0^{\pi/6} \cos^3(x)\,dx$

$\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right)$

$\cos^2(x) = \frac{1}{2}\left(\cos(2x) + 1\right)$

$\cos^3(x) = \frac{1}{2}\,\underline{\cos(2x)\cos(x)} + \frac{1}{2}\cos(x) = \frac{1}{4}\cos 3x + \frac{1}{4}\cos x + \frac{1}{2}\cos x$

$= \frac{1}{2}\left(\cos(3x) + \cos x\right)$

$= \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$

$$\int_0^{\pi/6} \cos^3 x\,dx = \frac{3}{4}\int_0^{\pi/6} \cos x\,dx + \frac{1}{4}\int_0^{\pi/6} \cos(3x)\,dx$$

$$= \frac{3}{4}\left[\sin x\right]_0^{\pi/6} + \frac{1}{4}\left[\frac{1}{3}\sin(3x)\right]_0^{\pi/6} = \frac{11}{24}$$

③ Calculer :

$t = \ln x = \varphi(x) \qquad \varphi'(x) = \dfrac{1}{x}$

a) $\displaystyle\int_1^e \frac{1}{x + x(\ln x)^2}\,dx$

$dt = dx \times \varphi'(x) = \dfrac{dx}{x}$

$\displaystyle = \int_1^e \frac{1}{1+(\ln x)^2} \times \frac{dx}{x} = \int_0^1 \frac{1}{1+t^2}\,dt = \left[\arctan x\right]_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

b) $\displaystyle\int_0^1 \frac{e^{2x}}{e^x + 1}\,dx \qquad t = e^x \qquad dt = dx\,\varphi'(x) = e^x dx$

$\displaystyle = \int_0^1 \frac{e^x}{e^x+1} \times e^x dx = \int_1^e \frac{t}{t+1}\,dt = \int_1^e \left(1 - \frac{1}{t+1}\right) dt$

$\displaystyle = \left[t - \ln(t+1)\right]_1^e = e - \ln(e+1) - 1 + \ln(2)$

Exercice bonus

Calculer $\int_0^{\pi/6} \frac{1}{\cos t}\, dt$ en posant $x = \sin t$

Solutions : martintroussard.fr