Nombres complexes (Exercices)

① Soit $z, z' \in \mathbb{C}$ tel que $|z| = |z'| = 1$. Montrer que $\dfrac{z+z'}{1+zz'} \in \mathbb{R}$

$$z = e^{i\theta} \qquad z' = e^{i\theta'}$$

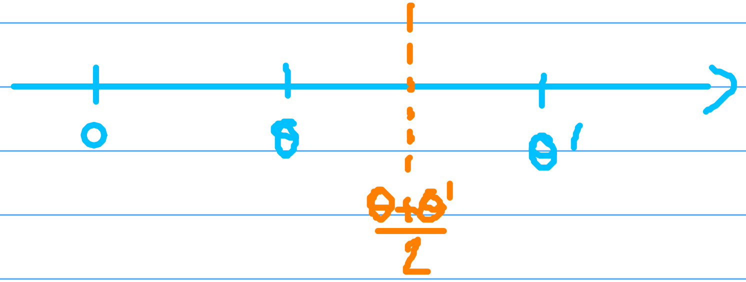
$$\frac{z+z'}{1+zz'} = \frac{e^{i\theta} + e^{i\theta'}}{1 + e^{i(\theta+\theta')}}$$

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{1+e^{-i(\theta+\theta')}}{(\cos(\theta+\theta')+1)^2 + (\sin(\theta+\theta'))^2}$$

$$= \frac{\cancel{e^{i\left(\frac{\theta+\theta'}{2}\right)}}\left(e^{i\left(\frac{\theta-\theta'}{2}\right)} + e^{i\left(\frac{\theta'-\theta}{2}\right)}\right)}{\cancel{e^{i\left(\frac{\theta+\theta'}{2}\right)}}\left(e^{-i\left(\frac{\theta+\theta'}{2}\right)} + e^{i\left(\frac{\theta+\theta'}{2}\right)}\right)}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$= \frac{\cancel{2}\cos\left(\frac{\theta-\theta'}{2}\right)}{\cancel{2}\cos\left(\frac{\theta+\theta'}{2}\right)}$$



② Déterminer $z \in \mathbb{C}^*$ tel que $|z| = \left|\frac{1}{z}\right| = |1-z|$

$z = r e^{i\theta}$ $\quad r \in \mathbb{R}_+^*$, $\theta \in]-\pi, \pi]$ $\quad \frac{1}{z} = \left(\frac{1}{r}\right) e^{-i\theta}$

Si z sol. $\quad r = \frac{1}{r} \Rightarrow r^2 = 1 \Rightarrow \boxed{r = 1}$

$z = e^{i\theta} = \cos\theta + i\sin\theta \qquad 1 - z = (1-\cos\theta) - i\sin\theta$

$|1-z|^2 = (1-\cos\theta)^2 + \sin^2\theta = 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta = 2 - 2\cos\theta$

Si z sol $\quad 2 - 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \boxed{\theta \in \left\{-\frac{\pi}{3}, \frac{\pi}{3}\right\}}$

On pose $z = e^{i\frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2} i$ ou $z = e^{-i\frac{\pi}{3}} = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

③ Transformer $\sin^5(x)$ et $\sin(5x)$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\sin^5 x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^5$$

$$= \frac{1}{32i}\left(e^{i5x} - 5e^{i3x} + 10e^{ix} - 10e^{-ix} + 5e^{-3ix} - e^{-5ix}\right)$$

$$= \frac{1}{16}\left(\frac{e^{5ix} - e^{-5ix}}{2i}\right) - \frac{5}{16}\left(\frac{e^{3ix} - e^{-3ix}}{2i}\right) + \frac{5}{8}\left(\frac{e^{ix} - e^{-ix}}{2i}\right)$$

$$= \frac{1}{16}\sin(5x) - \frac{5}{16}\sin(3x) + \frac{5}{8}\sin(x)$$

$$\sin(5x) = \operatorname{Im}\left((\cos x + i \sin x)^5\right)$$

$$= \operatorname{Im}\left(\cos^5 + 5i\cos^4\sin - 10\cos^3\sin^2 - 10i\cos^2\sin^3\right.$$

$$\left. + 5\cos\sin^4 + i\sin^5\right)$$

$$= 5\cos^4(x)\sin x - 10\cos^2(x)\sin^3(x) + \sin^5(x)$$

Exercice bonus

Soit $n \in \mathbb{N}$ et $\theta \in \mathbb{R}$. Mettre

$$(1 + e^{i\theta})^n$$

sous forme exponentielle.

Solutions : martintroussard.fr