Équations dans $\mathbb{C}$

Racines carrées

$$z^2 = \underset{\text{fixé}}{\underline{a}} \qquad a \in \mathbb{C}$$

$$(re^{i\theta})^2 = \underline{\rho}\, e^{i\underline{\omega}} \iff r^2 e^{2i\theta} = \rho\, e^{i\omega}$$

$r \geq 0 \qquad \rho \geq 0$

$\theta \in \mathbb{R} \qquad \omega \in \mathbb{R}$

$$\iff \begin{cases} r^2 = \rho \\ 2\theta \equiv \omega \ [2\pi] \end{cases} \qquad 2\theta = \omega + 2k\pi$$

$$\iff \begin{cases} r = \sqrt{\rho} \\ \theta \equiv \frac{\omega}{2} \ [\pi] \end{cases} \qquad \theta = \frac{\omega}{2} + k\pi$$

$$\iff z \in \left\{ \sqrt{\rho}\, e^{i\frac{\omega}{2}} ; \sqrt{\rho}\, e^{i\left(\frac{\omega}{2}+\pi\right)} \right\}$$

Exemple : $z^2 = 1+i = \sqrt{2}\, e^{i\frac{\pi}{4}} \iff z \in \left\{ \sqrt[4]{2}\, e^{i\frac{\pi}{8}}, \sqrt[4]{2}\, e^{i\left(\frac{\pi}{8}+\pi\right)} \right\}$

$z^2 = 5-12i$ $\qquad |5-12i| = \sqrt{25+144} = \sqrt{169} = 13$

$$5-12i = 13\left(\frac{5}{13} - \frac{12}{13} i\right)$$

$z = a+ib$ $\qquad z^2 = (a^2-b^2) + i(2ab)$ $\qquad |z^2| = |z|^2 = |5-12i|$

$$\begin{cases} a^2-b^2 = 5 \\ 2ab = -12 \\ a^2+b^2 = 13 \end{cases} \qquad \begin{matrix} 2a^2 = 18 \\ 2b^2 = 8 \end{matrix} \qquad \begin{matrix} a^2 = 9 \\ b^2 = 4 \end{matrix} \qquad \begin{matrix} a \in \{-3, 3\} \\ b \in \{-2, 2\} \end{matrix}$$

$2ab = -12 \Rightarrow ab = -6$ $\qquad \begin{cases} a = 3, b = -2 \\ a = -3, b = 2 \end{cases}$ $\qquad \begin{matrix} z = 3-2i \\ z = -3+2i \end{matrix}$ ou

Équations du second degré

Exemple : $2z^2 - (3-8i)z - (5+5i) = 0$

$\Delta = \left(-(3-8i)\right)^2 - 4 \times 2 \times \left(-(5+5i)\right) = 9 - 48i - 64 + 40 + 40i = -15 - 8i$

$(a+ib)^2 = -15 - 8i \iff \begin{cases} a^2 - b^2 = -15 \\ 2ab = -8 \end{cases} \Rightarrow \begin{cases} 2a^2 = 2 \\ 2b^2 = 32 \end{cases} \Rightarrow \begin{cases} a^2 = 1 \\ b^2 = 16 \end{cases}$

$|a+ib|^2 = |-15-8i| = \sqrt{225+64} = \sqrt{289} = 17 \qquad a^2 + b^2 = 17$

$a \in \{-1, 1\} \qquad b \in \{-4, 4\} \qquad ab = -4 \Rightarrow \begin{cases} a = 1,\ b = -4 & 1-4i \\ a = -1,\ b = 4 & -1+4i \end{cases}$

$z_1 = \dfrac{(3-8i) - (1-4i)}{4} = \dfrac{1}{2} - i \qquad z_2 = \dfrac{(3-8i) + (1-4i)}{4} = 1 - 3i$

Racines n-ièmes de l'unité

$n \in \mathbb{N}^*$

$$z^n = 1 \iff z \in \left\{ e^{i\frac{2k\pi}{n}} \;\middle|\; k \in \{0, \dots, n-1\} \right\}$$

$z = re^{i\theta}$ $\quad r \in \mathbb{R}_+ \quad \theta \in [0, 2\pi[$

$$z^n = 1 \iff r^n e^{in\theta} = 1 \iff \begin{cases} r^n = 1 \\ n\theta \equiv 0 \ [2\pi] \end{cases} \iff \begin{cases} r = 1 \\ \theta = \frac{2k\pi}{n}, \end{cases}$$

$n\theta = 2k\pi \quad (k \in \mathbb{Z})$

$k \in \{0, \dots, n-1\}$

$$0 \leq \frac{2k\pi}{n} < 2\pi \iff 0 \leq k < n$$

Racines n-ième d'un nombre complexe → Retour à l'unité

Exemple : $z^3 = 8 \iff z^3 = 2^3$

$\iff \left(\frac{z}{2}\right)^3 = 1 \iff \frac{z}{2} \in \{1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}\}$

$\iff z \in \{2, 2e^{i\frac{2\pi}{3}}, 2e^{i\frac{4\pi}{3}}\}$

$z^5 = i \iff z^5 = i^5 \iff \left(\frac{z}{i}\right)^5 = 1$

$\iff \frac{z}{i} \in \{1, e^{i\frac{2\pi}{5}}, \dots\} \iff z \in \{i, ie^{i\frac{2\pi}{5}}, ie^{i\frac{4\pi}{5}}, \dots\}$

⚠ Cas particulier

À vous de jouer !

① Résoudre :

a) $z^2 = 24 + 10i$

b) $z^2 - (2+7i)z - (3-6i) = 0$

c) $z^3 = i$

② Soit $n \in \mathbb{N}$. Résoudre $(z-i)^n = (z+i)^n$