Équations dans $\mathbb{C}$ (Exercices)

① Résoudre :

a) $z^2 = 24 + 10i$ $\qquad |24 + 10i| = \sqrt{576 + 100} = \sqrt{676} = 26$

$\underbrace{(a+ib)^2}_{a^2 + 2iab - b^2} = 24 + 10i \iff \begin{cases} a^2 - b^2 = 24 \\ 2ab = 10 \end{cases} \implies \begin{cases} 2a^2 = 50 \\ 2b^2 = 2 \end{cases} \implies \begin{cases} a^2 = 25 \\ b^2 = 1 \end{cases}$

$|z^2| = |24 + 10i|$

$|z|^2 = a^2 + b^2 = 26$

$a \in \{-5, 5\} \qquad b \in \{-1, 1\}$

$2ab = 10 \iff ab = 5 \implies \begin{cases} a = 5, b = 1 \\ a = -5, b = -1 \end{cases} \implies \begin{cases} z = 5 + i \text{ ou} \\ z = -5 - i \end{cases}$

b) $2z^2-(2+7i)z-(3-6i)=0$

$\Delta=(-(2+7i))^2-4\times 2\times(-(3-6i))=-21-20i$ " $\sqrt{\Delta}$ " $=\begin{cases}2-5i\\-2+5i\end{cases}$

$(a+ib)^2=-21-20i \iff \begin{cases}a^2-b^2=-21\\2ab=-20\\a^2+b^2=29\end{cases} \implies \begin{cases}2a^2=8\\2b^2=50\end{cases} \implies \begin{cases}a^2=4\\b^2=25\end{cases}$

$|a+ib|^2=|-21-20i|=\sqrt{441+400}=\sqrt{841}=29$

$a\in\{-2,2\}$, $b\in\{-5,5\}$ $ab=-10 \implies \begin{cases}a=2, b=-5\\a=-2, b=5\end{cases}$

$z_1=\dfrac{(2+7i)-(2-5i)}{4}=3i$ $z_2=\dfrac{(2+7i)+(2-5i)}{4}=1+\dfrac{1}{2}i$

c) $z^3=i \iff z^3=(-i)^3 \iff \left(\dfrac{z}{-i}\right)^3=1$

$\iff z\in\left[-i, -ie^{\frac{2i\pi}{3}}, -ie^{\frac{4i\pi}{3}}\right]$

② Soit $n \in \mathbb{N}^*$. Résoudre $(z-i)^n = (z+i)^n$

$z = i$ $\quad 0^n = (2i)^n$ Pas une sol. Soit $z \neq i$

$\left(\frac{z+i}{z-i}\right)^n = 1$ $\quad \frac{z+i}{z-i} = e^{i\frac{2k\pi}{n}}, k \in \{0, \dots, n-1\}$

$z+i = e^{\cdots}(z-i) \iff z(1-e^{\cdots}) = -i - e^{\cdots} i$

$\iff z = \frac{-i - e^{\cdots} i}{1-e^{\cdots}} = \frac{i(1+e^{\cdots})}{e^{\cdots}-1}$

$|z-i| = |z+i| \iff |r\cos\theta + i(r\sin\theta - 1)| = |r\cos\theta + i(r\sin\theta + 1)|$

$\iff \cancel{r^2\cos^2} + \cancel{r^2\sin^2} - 2r\sin + \cancel{1} = \cancel{r^2\cos^2} + \cancel{r^2\sin^2} + 2r\sin + \cancel{1}$

$\iff 4r\sin\theta = 0$ $\quad \sin\theta = 0 \Rightarrow z \in \mathbb{R}$

Exercice bonus

Résoudre dans $\mathbb{C}^2$:

$$\begin{cases} xy = -16+4i \\ x-y = 7+i \end{cases}$$

Solutions : martintrousard.fr