Trigonométrie hyperbolique

$$e^{ix} = \cos x + i \sin x$$

Fonctions hyperboliques

Motivation

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\operatorname{ch} x = \frac{e^{x} + e^{-x}}{2} \qquad \operatorname{sh} x = \frac{e^{x} - e^{-x}}{2} \qquad \operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x}$$

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{paire}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{impaire}} \qquad e^{x} = \underbrace{\operatorname{ch} x}_{\text{paire}} + \underbrace{\operatorname{sh} x}_{\text{impaire}}$$

Propriété fondamentale

$$\operatorname{ch}^2 x - \operatorname{sh}^2 x = (\operatorname{ch} x + \operatorname{sh} x)(\operatorname{ch} x - \operatorname{sh} x) = e^{x} e^{-x} = 1$$

Dérivée

$\cos' = -\sin$, $\sin' = \cos$, $\tan' = \dfrac{1}{\cos^2} = 1 + \tan^2$

$\mathrm{ch} : x \longmapsto \dfrac{e^x + e^{-x}}{2}$ $\quad \mathrm{ch}' = \mathrm{sh}$ $\qquad \mathrm{sh} : x \longmapsto \dfrac{e^x - e^{-x}}{2}$ $\quad \mathrm{sh}' = \mathrm{ch}$

$\mathrm{th} : x \longmapsto \dfrac{\mathrm{sh}\, x}{\mathrm{ch}\, x}$ $\qquad \mathrm{th}'(x) = \dfrac{\mathrm{ch}\, x \times \mathrm{ch}\, x - \mathrm{sh}\, x \times \mathrm{sh}\, x}{\mathrm{ch}^2 x} = \dfrac{1}{\mathrm{ch}^2 x} = 1 - \mathrm{th}^2 x$

$\mathrm{th}\, x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x} - 1}{e^{2x} + 1}$

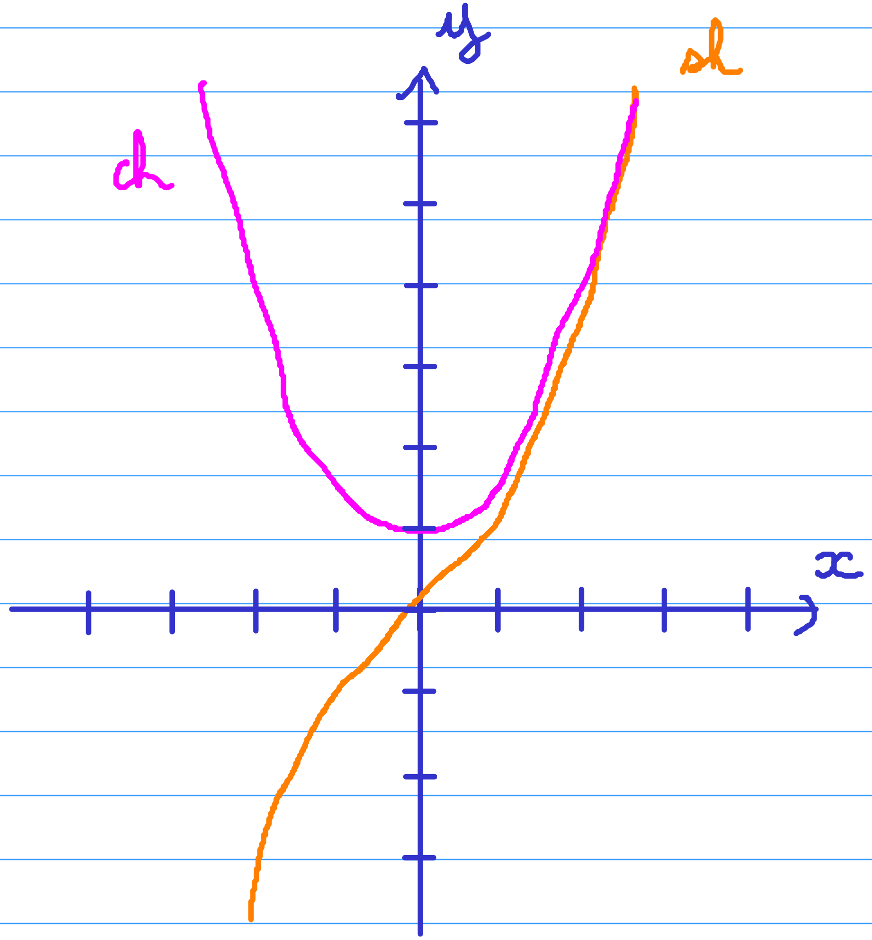
x	$-\infty$		0		$+\infty$
$\mathrm{ch}\, x$	$+\infty$	↘	1	↗	$+\infty$

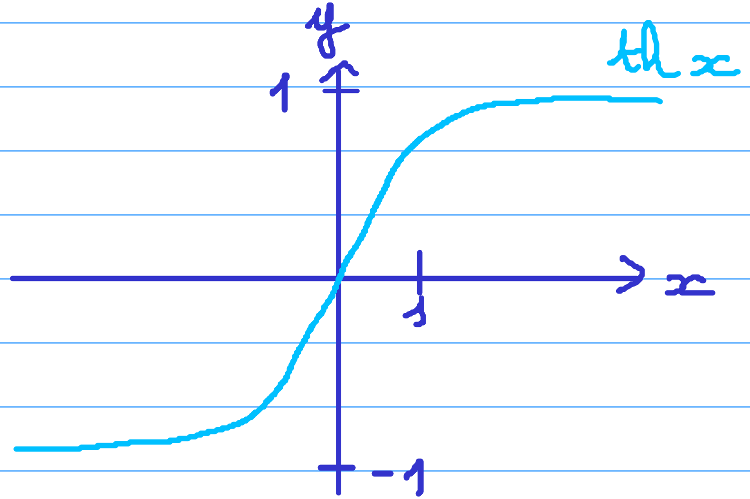
x	$-\infty$		0		$+\infty$
$\mathrm{sh}\, x$	$-\infty$	↗	0	↗	$+\infty$

x	$-\infty$		$+\infty$
$\mathrm{th}\, x$	-1	↗	1

$\mathbb{R} \to]-1, 1[$

Courbes représentatives





Formules d'addition

$$\operatorname{ch}(a+b) = \frac{e^{a+b} + e^{-a-b}}{2}$$

$$\operatorname{ch} a \operatorname{ch} b = \frac{1}{4}\left(e^{a+b} + \underline{e^{a-b} + e^{-a+b}} + e^{-a-b}\right)$$

$$\operatorname{sh} a \operatorname{sh} b = \frac{1}{4}\left(e^{a+b} \underline{- e^{a-b} - e^{-a+b}} + e^{-a-b}\right)$$

$$\operatorname{ch}(a+b) = \operatorname{ch} a \operatorname{ch} b + \operatorname{sh} a \operatorname{sh} b$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} = \operatorname{ch}(ix) \qquad \cos(ix) = \operatorname{ch}(i \times ix) = \operatorname{ch}(-x) = \operatorname{ch} x$$

$$\sin x = \frac{\operatorname{sh}(ix)}{i} \qquad \sin(ix) = \frac{\operatorname{sh}(i \times ix)}{i} = \frac{\operatorname{sh}(-x)}{i} = \frac{-\operatorname{sh} x}{i} = i \operatorname{sh} x$$

$$\operatorname{ch}(a+b) = \cos(ia + ib) = \underbrace{\cos(ia)}_{\operatorname{ch} a} \underbrace{\cos(ib)}_{\operatorname{ch} b} - \underbrace{\sin(ia)}_{i \operatorname{sh} a} \underbrace{\sin(ib)}_{i \operatorname{sh} b}$$

À vous de jouer !

① Soit $x \in \mathbb{R}$, donner une formule pour :

$$\mathrm{ch}(2x), \quad \mathrm{sh}(2x), \quad \mathrm{th}(2x)$$

$\mathrm{sh}(a+b)$

② Soit $a, b \in \mathbb{R}_+^*$, montrer qu'il existe $(\alpha, \beta) \in \mathbb{R}^2$:

$$\forall x \in \mathbb{R}, \quad a e^{x} + b e^{-x} = \alpha \, \mathrm{ch}(x+\beta)$$

$$a \cos x + b \sin x = A \cos(x + \varphi)$$