Trigonométrie hyperbolique (Exercices)

$\cos(2x) = \cos^2 x - \sin^2 x$
$= 1 - 2\sin^2 x$
$= 2\cos^2 x - 1$

① Soit $x \in \mathbb{R}$, donner une formule pour :

$\operatorname{ch}(2x) = \operatorname{ch}(x+x) = \operatorname{ch}x \operatorname{ch}x + \operatorname{sh}x \operatorname{sh}x = \operatorname{ch}^2 x + \operatorname{sh}^2 x$

$= 1 + 2\operatorname{sh}^2 x = 2\operatorname{ch}^2 x - 1$

$\cos(x) = \operatorname{ch}(ix) \Rightarrow \cos(ix) = \operatorname{ch}(-x) = \operatorname{ch} x$ $\quad \sin(x) = \frac{1}{i}\operatorname{sh}(ix)$

$\sin(ix) = -\frac{1}{i}\operatorname{sh}(x) = i\operatorname{sh}x$

$$\operatorname{sh}(a+b) = \frac{\sin(ia+ib)}{i} = \frac{\overbrace{\cos(ia)}^{\operatorname{ch}a}\overbrace{\sin(ib)}^{i\operatorname{sh}b} + \overbrace{\sin(ia)}^{i\operatorname{sh}a}\overbrace{\cos(ib)}^{\operatorname{ch}b}}{i} = \operatorname{ch}a \operatorname{sh}b + \operatorname{sh}a \operatorname{ch}b$$

$\operatorname{sh}(2x) = \operatorname{sh}(x+x) = 2\operatorname{ch}x \operatorname{sh}x$

$$\operatorname{th}(2x) = \frac{2\operatorname{ch}x\operatorname{sh}x}{\operatorname{ch}^2 x + \operatorname{sh}^2 x} = \frac{2\frac{\operatorname{sh}x}{\operatorname{ch}x}}{1 + \frac{\operatorname{sh}^2 x}{\operatorname{ch}^2 x}} = \frac{2\operatorname{th}x}{1+\operatorname{th}^2 x}$$

② Soit $a, b \in \mathbb{R}_+^*$, montrer qu'il existe $(\alpha, \beta) \in \mathbb{R}^2$:

$$\forall x \in \mathbb{R}, \quad \underbrace{a e^{x}}_{>0} + \underbrace{b e^{-x}}_{>0} = \underbrace{\alpha}_{>0} \underbrace{ch(x+\beta)}_{>0}$$

Analyse $x \in \mathbb{R}$, $\quad a e^{x} + b e^{-x} = a(ch\, x + sh\, x) + b(ch(-x) + sh(-x))$

$$= (a+b)\, ch\, x + (a-b)\, sh\, x$$

$$\alpha\, ch(x+\beta) = \alpha\, ch\beta\, ch\, x + \alpha\, sh\beta\, sh\, x$$

$$\begin{cases} a+b = \alpha\, ch\beta \\ a-b = \alpha\, sh\beta \end{cases}$$

$ch^2\beta - sh^2\beta = 1$

$\alpha^2 ch^2\beta - \alpha^2 sh^2\beta = \alpha^2 = (a+b)^2 - (a-b)^2$

$\alpha^2 = 4ab$

$\alpha = 2\sqrt{ab}$

$$\begin{cases} a+b = 2\sqrt{ab}\, ch\beta \\ a-b = 2\sqrt{ab}\, sh\beta \end{cases} \Rightarrow \cancel{2}a = \cancel{2}\sqrt{ab}\, e^{\beta} \Rightarrow e^{\beta} = \sqrt{\frac{a}{b}} \Rightarrow \beta = \ln\left(\sqrt{\frac{a}{b}}\right)$$

Synth. $ch\beta = \dfrac{e^{\beta}+e^{-\beta}}{2} = \dfrac{\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}}{2}$

$\alpha\, ch\beta = a+b \qquad \alpha\, sh\beta = a-b$

$\alpha\, ch(x+\beta) = (a+b)\, ch\, x + (a-b)\, sh\, x = a e^{x} + b e^{-x}$

Exercice bonus

Soit $n \in \mathbb{N}$ et $x \in \mathbb{R}$. Calculer :

$$\prod_{k=1}^{n} \operatorname{ch}\left(\frac{x}{2^k}\right)$$

Solutions : martintroussard.fr