Fonctions hyperboliques réciproques (Exercices)

$[1,+\infty[$

① Résoudre dans $\cancel{\mathbb{R}}$: $\operatorname{argch} x + \operatorname{argsh} x = 1$

<u>Analyse</u> : $x \in \mathbb{R}$

$$\operatorname{argch} x + \operatorname{argsh} x = 1$$

$$\operatorname{ch}(\operatorname{argch} x + \operatorname{argsh} x) = \operatorname{ch}(1)$$

$\operatorname{ch}(a+b) = \operatorname{ch} a \operatorname{ch} b + \operatorname{sh} a \operatorname{sh} b$

$$x \operatorname{ch}(\operatorname{argsh} x) + \operatorname{sh}(\operatorname{argch} x)\, x = \operatorname{ch}(1)$$

$$\operatorname{ch}(\operatorname{argsh} x) = \sqrt{\operatorname{ch}^2(\operatorname{argsh} x)} = \sqrt{1+\operatorname{sh}^2(\operatorname{argsh} x)} = \sqrt{1+x^2}$$

$$\underbrace{\operatorname{sh}(\underbrace{\operatorname{argch} x}_{>0})}_{>0} = \sqrt{x^2-1}$$

$$x\left(\sqrt{x^2+1} + \sqrt{x^2-1}\right) = \operatorname{ch}(1)$$

$$x\left(\underline{\sqrt{x^2+1}+\sqrt{x^2-1}}\right)=\mathrm{ch}(1)$$

$$\mathrm{argch}\,x+\mathrm{argsh}\,x=1 \iff \mathrm{sh}(\mathrm{argch}\,x+\mathrm{argsh}\,x)=\mathrm{sh}(1)$$

$\mathrm{sh}(a+b)=\mathrm{ch}\,a\,\mathrm{sh}\,b+\mathrm{sh}\,a\,\mathrm{ch}\,b$

$$\iff x\times x+\mathrm{sh}(\mathrm{argch}\,x)\,\mathrm{ch}(\mathrm{argsh}\,x)=\mathrm{sh}(1)$$

$$\iff x^2+\underline{\sqrt{x^2-1}\,\sqrt{x^2+1}}=\mathrm{sh}(1)$$

$$x^2\left(x^2+1+2\sqrt{x^2-1}\,\sqrt{x^2+1}+x^2-1\right)=\mathrm{ch}^2(1)$$

$$x^2\left(2x^2+2\sqrt{x^2-1}\,\sqrt{x^2+1}\right)=\mathrm{ch}^2(1)$$

$$x^2\times 2\,\mathrm{sh}(1)=\mathrm{ch}^2(1) \iff x^2=\frac{\mathrm{ch}^2(1)}{2\,\mathrm{sh}(1)} \iff x=\frac{\mathrm{ch}(1)}{\sqrt{2\,\mathrm{sh}(1)}}$$

<u>Synthèse</u> : Vérifier que $x \geq 1$

$$\operatorname{argch}(x) = \ln\left(x+\sqrt{x^2-1}\right)$$

② Simplifier :

a) $\operatorname{argch}\left(\sqrt{\dfrac{1+x}{2}}\right) = \ln\left(\sqrt{\dfrac{1+x}{2}} + \sqrt{\sqrt{\dfrac{1+x}{2}}^{2} - 1}\right)$

$$= \ln\left(\sqrt{\frac{1+x}{2}} + \sqrt{\frac{x-1}{2}}\right) = \ln\left(\frac{1}{\sqrt{2}}\left[\sqrt{1+x}+\sqrt{x-1}\right]\right)$$

$$\left(\sqrt{1+x}+\sqrt{x-1}\right)^2 = 1+x+2\sqrt{x^2-1}+x-1 = 2x+2\sqrt{x^2-1}$$

$$\sqrt{1+x}+\sqrt{x-1} = \sqrt{2}\sqrt{x+\sqrt{x^2-1}}$$

$$= \ln\left(\sqrt{x+\sqrt{x^2-1}}\right) = \frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right) = \frac{1}{2}\operatorname{argch} x$$

b) $\operatorname{argth}\left(\dfrac{1+2\operatorname{th}x}{2+\operatorname{th}x}\right)$

$$\operatorname{th}(a+b) = \frac{\operatorname{sh}(a+b)}{\operatorname{ch}(a+b)} = \frac{\operatorname{ch}a\operatorname{sh}b+\operatorname{sh}a\operatorname{ch}b}{\operatorname{ch}a\operatorname{ch}b+\operatorname{sh}a\operatorname{sh}b} = \frac{\dfrac{\operatorname{sh}b}{\operatorname{ch}b}+\dfrac{\operatorname{sh}a}{\operatorname{ch}a}}{1+\dfrac{\operatorname{sh}a}{\operatorname{ch}a}\times\dfrac{\operatorname{sh}b}{\operatorname{ch}b}}$$

$$= \frac{\operatorname{th}a+\operatorname{th}b}{1+\operatorname{th}a\operatorname{th}b}$$

$\frac{1}{2} \in]-1,1[$

$$= \operatorname{argth}\left(\frac{\frac{1}{2}+\operatorname{th}x}{1+\frac{1}{2}\times\operatorname{th}x}\right)$$

$a \in \mathbb{R}$ tq $\operatorname{th}(a) = \dfrac{1}{2}$ $\quad a = \operatorname{argth}\left(\dfrac{1}{2}\right)$

$$= \operatorname{argth}(\operatorname{th}(x+a)) = x+a = x+\operatorname{argth}\left(\frac{1}{2}\right) = x+\ln(3)$$

$$\operatorname{argth}\left(\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \frac{1}{2}\ln(3)$$

Exercice bonus

Trouver une primitive de :

a) $f : x \longmapsto \dfrac{1}{\sqrt{x^2+a^2}}$

b) $f : x \longmapsto \dfrac{1}{\sqrt{x^2-a^2}}$

c) $f : x \longmapsto \dfrac{1}{\sqrt{a^2-x^2}}$

Solutions : martintroussard.fr