Matrices

Définitions et notations

Matrice à n lignes et p colonnes

$\mathcal{M}_{n,p}(\mathbb{R})$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}$$

$$A = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}}$$

$$(ij)_{\substack{1 \leq i \leq 5 \\ 1 \leq j \leq 10}} \qquad \begin{array}{cc} \overbrace{\begin{pmatrix} 1 & 2 & \cdots & 10 \\ 2 & 4 & \cdots & 20 \\ \vdots & \vdots & & \vdots \\ 5 & 10 & \cdots & 50 \end{pmatrix}}^{10} & \Big\} 5 \end{array}$$

Matrices carrées

$\mathcal{M}_n(\mathbb{R})$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$A = (a_{ij})_{1 \leq i,j \leq n}$$

Matrices particulières

Identité

$$I_n = \begin{pmatrix} 1 & & (0) \\ & \ddots & \\ (0) & & 1 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R})$$

Matrice nulle

$$0_{n,p} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in \mathcal{M}_{n,p}(\mathbb{R}) \qquad 0_n = \begin{pmatrix} 0 & & (0) \\ & \ddots & \\ (0) & & 0 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R})$$

Matrices colonnes | lignes

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{M}_{n,1}(\mathbb{R}) \qquad X = (x_1 \cdots x_n) \in \mathcal{M}_{1,n}(\mathbb{R})$$

Transposée

$$A = \begin{array}{cc} p & \\ \begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{np} \end{pmatrix} & n \end{array} \qquad {}^tA = \begin{array}{cc} n & \\ \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1p} & \cdots & a_{np} \end{pmatrix} & p \end{array}$$

A^T

Produit matriciel

m̂ dim.

$$\begin{array}{cc} & \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} & \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & \times \\ \times & a_{21}b_{12}+a_{22}b_{22} \\ \times & \times \end{pmatrix} \end{array}$$

⚠ Dimensions

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}$$

$$AB = \begin{array}{cc} & \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \end{pmatrix} \end{array}$$

$$\cancel{BA} = \begin{array}{cc} \neq & \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix} & \end{array}$$

Cas général

$A \in \mathcal{M}_{n,p}(\mathbb{R})$

$B \in \mathcal{M}_{p,q}(\mathbb{R})$

$AB \in \mathcal{M}_{n,q}(\mathbb{R})$

$$\begin{array}{c} j \\ \begin{pmatrix} b_{11} & b_{12} & b_{1j} & \cdot & b_{1q} \\ b_{21} & b_{22} & b_{2j} & \cdot & b_{2q} \\ \vdots & \vdots & & & \vdots \\ b_{p1} & b_{p2} & b_{pj} & & b_{pq} \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ & & & \end{pmatrix} \\ i & \begin{pmatrix} a_{i1} & a_{i2} & & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix} \end{array}$$

$$\boxed{\sum_{k=1}^{p} a_{ik} b_{kj}}$$

⚠ Commutativité

$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$AB \neq BA$

⚠ Binôme

$$(A+B)^2 = A^2 + AB + BA + B^2$$

$\neq 2AB$

Applications linéaires

$f : E \to E \qquad E = \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots$

$\forall (x,y) \in E, \forall \lambda \in \mathbb{R}, \quad \underline{f(\lambda x + y) = \lambda f(x) + f(y)}$

$f(\lambda x) = \lambda f(x)$
$f(x+y) = f(x) + f(y)$

$\underline{E = \mathbb{R}}$ $\quad \lambda \in \mathbb{R}, f(\lambda) = f(\lambda \times 1) = \lambda \underbrace{f(1)}_{\text{const.}} \qquad f : x \longmapsto ax \quad (a \in \mathbb{R})$

$\underline{E = \mathbb{R}^2}$ $\quad f((a,b)) = f(a \times (1,0) + b \times (0,1)) = a \underbrace{f((1,0))}_{\in \mathbb{R}^2} + b \underbrace{f((0,1))}_{\in \mathbb{R}^2}$

$\underline{\text{Ex}}$: $f((1,0)) = (-3,7) \qquad f((0,1)) = (2,5)$

$f(a,b) = (-3a+2b \,;\, 7a+5b) \qquad \begin{pmatrix} a \\ b \end{pmatrix}$

$A = \begin{pmatrix} -3 & 2 \\ 7 & 5 \end{pmatrix} \qquad X = \begin{pmatrix} a \\ b \end{pmatrix} \qquad AX = \begin{pmatrix} -3 & 2 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} -3a+2b \\ 7a+5b \end{pmatrix} = f((a,b))$

$A : f \qquad B : g \qquad f \circ g(x) = A(BX) = (AB)X$

À vous de jouer !

① Déterminer l'ensemble des

a) $B \in \mathcal{M}_2(\mathbb{R})$ commutant avec $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

b) $B \in \mathcal{M}_n(\mathbb{R})$ commutant avec $A = \begin{pmatrix} 1 & & & (0) \\ & 2 & & \\ & & \ddots & \\ (0) & & & n \end{pmatrix}$

② Soit $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Calculer $(A - I_3)^n$, puis A^n.