Matrices (Exercices)

① Déterminer l'ensemble des

a) $B \in \mathcal{M}_2(\mathbb{R})$ commutant avec $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad a, b, c, d \in \mathbb{R}$

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \qquad BA = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

$$\begin{pmatrix} b & a \\ a & b \end{pmatrix} \qquad \begin{pmatrix} b & a \\ a & b \end{pmatrix}$$

$AB = BA \iff \begin{cases} a = d \\ b = c \end{cases}$ $\qquad \underline{B = \begin{pmatrix} a & b \\ b & a \end{pmatrix}}$

$$\mathcal{S} = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \,\middle|\, (a,b) \in \mathbb{R}^2 \right\} = \left\{ a \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,\middle|\, (a,b) \in \mathbb{R}^2 \right\}$$

$$= \left\{ a I_2 + bA \,\middle|\, (a,b) \in \mathbb{R}^2 \right\}$$

b) $B \in \mathcal{M}_n(\mathbb{R})$ commutant avec $A = \begin{pmatrix} 1 & & & (0) \\ & 2 & & \\ & & \ddots & \\ (0) & & & n \end{pmatrix}$ $\qquad B = (b_{ij})_{1\leq i,j\leq n}$

$$\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & & b_{nn} \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & & & (0) \\ & 2 & & \\ & & \ddots & \\ (0) & & & n \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ 2b_{21} & 2b_{22} & \dots & 2b_{2n} \\ \vdots & \vdots & & \\ nb_{n1} & nb_{n2} & \dots & nb_{nn} \end{pmatrix} \qquad AB = (i\,b_{ij})_{1\leq i,j\leq n}$$

$$\begin{pmatrix} 1 & & & (0) \\ & 2 & & \\ & & \ddots & \\ (0) & & & n \end{pmatrix}$$

$$BA = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & & b_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & 2b_{12} & \dots & n\,b_{1n} \\ b_{21} & 2b_{22} & \dots & n\,b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & 2b_{n2} & \dots & n\,b_{nn} \end{pmatrix} \qquad BA = (j\,b_{ij})_{1\leq i,j\leq n}$$

$$AB = BA \iff \forall (i,j) \in \{1, \dots, n\}^2, \quad i\, b_{ij} = j\, b_{ij}$$

$$(i-j)\, b_{ij} = 0$$

$$i \neq j \implies b_{ij} = 0$$

$$B = \begin{pmatrix} b_{11} & & (0) \\ & \ddots & \\ (0) & & b_{nn} \end{pmatrix}$$

$$AB = \begin{pmatrix} b_{11} & & & (0) \\ & 2b_{22} & & \\ & & \ddots & \\ (0) & & & n\, b_{nn} \end{pmatrix} = BA$$

$$\mathcal{C} = \left\{ \begin{pmatrix} \lambda_1 & & & (0) \\ & \lambda_2 & & \\ & & \ddots & \\ (0) & & & \lambda_n \end{pmatrix} \;\middle|\; (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \right\}$$

② Soit $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Calculer $(A-I_3)^n$, puis A^n.

$$A - I_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(A-I_3)^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(A-I_3)^3 = (A-I_3)^2(A-I_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\forall n \geqslant 3, \ (A-I_3)^n = 0$

$$A^n = \left[(A - I_3) + I_3\right]^n$$

$$AI_3 = I_3 A = A$$

$$= \sum_{k=0}^{n} \binom{n}{k} (A - I_3)^n \, I_3^{\,n-k}$$

$$I_3^{\,n-k} = I_3$$

$\underline{n \geqslant 2}$

$$= \sum_{k=0}^{n} \binom{n}{k} (A - I_3)^n = \sum_{k=0}^{2} \binom{n}{k} (A - I_3)^k$$

$$= I_3 + n(A - I_3) + \binom{n}{2} (A - I_3)^2$$

$$\binom{n}{2} = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + n \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{n(n-1)}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^n = \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \quad n \geqslant 2$$

$n = 0$ et $n = 1$ ✓

Exercice bonus

Soit $\theta \in \mathbb{R}$ et $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Calculer A^n.

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