Variables aléatoires (Exercices)

$$
\begin{array}{lllll}
k=0 & P(1) + & P(2) + \cdots + & & P(n) \\
k=1 & 1P(1) & P(2) + \cdots + & & P(n) \\
k=2 & 0 & 2P(2) & P(3) + \cdots + & P(n) \\
\vdots & & & & \\
k=n-1 & 0 & 0 & 0 & n\,P(n)
\end{array}
$$

1) Soit $X : \Omega \to \{1, \ldots, n\}$

Montrer que $\mathbb{E}(X) = \sum_{k=0}^{n-1} P(X > k)$

$$\mathbb{E}(X) = \sum_{k=1} k\, P(X = k)$$

$(X > k-1) = (X = k) \cup (X > k)$ disjoint

$$P(X > k-1) = P(X = k) + P(X > k)$$

$$P(X = k) = P(X > k-1) - P(X > k)$$

$$\mathbb{E}(X) = \sum_{k=1}^{n} k\, P(X = k) = \sum_{k=1}^{n} k \left[P(X > k-1) - P(X > k) \right]$$

$$= \sum_{k=1}^{n} k\, P(X > k-1) - \sum_{k=1}^{n} k\, P(X > k)$$

$$= \sum_{k=0}^{n-1} (k+1)\, P(X > k) - \sum_{k=0}^{n-1} k\, P(X > k) = \sum_{k=0}^{n-1} P(X > k)$$

② Soit X, Y deux variables aléatoires à valeurs dans $\{1, \dots, n\}$ tel que $\forall i \in \{1, \dots, n\}$, $P(X = i) = P(Y = i) = \dfrac{1}{n}$

$n = 6$

On suppose $\forall\ 1 \leq i, j \leq n$, $(X = i)$ et $(Y = j)$ indépendants

Donner la loi de probabilité de $X + Y$.

disjointe

$i \in \{2, \dots, 2n\}$ $\qquad (X+Y = i) = \bigcup_{k=1}^{i-1} \left[(X = k) \cap (Y = i-k)\right]$

$$P(X+Y = i) = \sum_{k=1}^{i-1} P\big((X=k) \cap (Y = i-k)\big)$$

ind.

$$= \sum_{k=1}^{i-1} P(X = k)\, P(Y = i-k)$$

$$\forall i \in \{2,\dots,2n\}, \quad P(X+Y=i) = \sum_{k=1}^{i-1} P(X=k)\,P(Y=i-k)$$

$i = 2n \qquad P(X+Y=2n) = \sum_{k=1}^{2n-1} P(X=k)\,P(Y=2n-k)$

$(n=6) \qquad 12$

$P(X=1)\,\underbrace{P(Y=11)}_{=0} \qquad k=1$

$\underbrace{P(X=11)}_{=0}\,P(Y=1) \qquad k=11$

$P(X=k) \neq 0 \iff k \leq n$

$P(X=i-k) \neq 0 \iff i-k \leq n \iff k \geq i-n$

- $\underline{i \leq n}$ $\qquad k \leq i-1 < n \qquad i-n \leq 0 < k$

$$P(X+Y=i) = \sum_{k=1}^{i-1} \frac{1}{n} \times \frac{1}{n} = \frac{i-1}{n^2}$$

- $\underline{i > n}$

$$P(X+Y=i) = \sum_{k=i-n}^{n} \frac{1}{n} \times \frac{1}{n} = \frac{2n-i+1}{n^2}$$

Exercice bonus

Montrer que $\mathrm{Cov}(X,Y) = \mathbb{E}\big((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\big)$

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